# Algebraic models of dependent type theory

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These slides: `https://goo.gl/Ttacdq`

# Representable natural transformations

A map $f : Y \to X$ in $\widehat{\mathbb{C}} := [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is **representable** if

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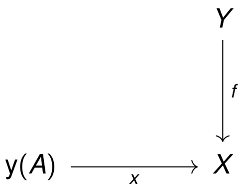

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there exist $g : B \to A$ in $\mathbb{C}$ and $y \in Y(B)$
such that the following square is a pullback:

$$\begin{array}{ccc} \mathrm{y}(B) & \xrightarrow{\;y\;} & Y \\ {\scriptstyle \mathrm{y}(g)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{y}(A) & \xrightarrow[\;x\;]{} & X \end{array}$$

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  $\forall \Gamma, A \; \exists$ (chosen) $\Gamma \centerdot A, \mathsf{p}_A, \mathsf{q}_A$

$$\begin{array}{ccc} \mathsf{y}(\Gamma \centerdot A) & \xrightarrow{\mathsf{q}_A} & \dot{\mathcal{U}} \\ {\scriptstyle \mathsf{y}(\mathsf{p}_A)} \downarrow & \lrcorner & \downarrow {\scriptstyle p} \\ \mathsf{y}(\Gamma) & \xrightarrow[A]{} & \mathcal{U} \end{array}$$

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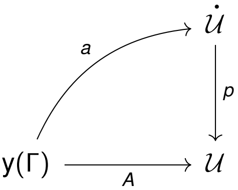

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Officially, $P_f$ is the composite

$$\mathcal{E} \xrightarrow{\Delta_{B \to 1}} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_{A \to 1}} \mathcal{E}$$

where $\Delta_f$ is pullback along $f$ and $\Sigma_f \dashv \Delta_f \dashv \Pi_f$.

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- *Polynomial functors and cartesian natural transformations are the 1- and 2-cells of a **2-category**;*
- *These are biequivalent.*

# Admitting a unit type

## Theorem (Awodey)

*A natural model admits a unit type ⇔ there exist $\widehat{\mathbf{1}}$, $\widehat{\star}$ as in:*

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## Corollary

*interpretations of unit types* $\leftrightsquigarrow$ *cartesian morphisms of polynomials* $1 \Rightarrow p$

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*A natural model admits a* $\Pi$*-types* $\Leftrightarrow$ *there exist* $\widehat{\Pi}$, $\widehat{\lambda}$ *as in:*

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| *interpretations of* $\Pi$*-types* | $\leftrightsquigarrow$ | *cartesian morphisms of polynomials* $P_p(p) \Rightarrow p$ |

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**Goal.** Find the appropriate notion of 3-cell (morphism *of morphisms* of polynomials) allowing us to make this more precise.

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**Idea:** Given cartesian morphisms $\varphi, \psi : f \Rightarrow g$, take internal natural transformations $\mathbb{S}(\varphi) \Rightarrow \mathbb{S}(\psi)$ to be our 3-cells.

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**Bonus:** If $(\mathbb{C}, p)$ admits $1, \Sigma, \Pi$, then $\mathbb{U}$ is cartesian closed.

# Initiality of the syntax

## Idea (Initiality 'conjecture')

*The syntax of a dependent type theory $\mathbb{T}$ should itself have the structure of a natural model, which is initial amongst all natural models interpreting $\mathbb{T}$.*

# Initiality of the syntax

## Idea (Initiality 'conjecture')

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**Goals:**

- Build the syntactic natural models for some basic type theories and prove that they satisfy an appropriate universal property;
- Expand to more complicated type theories by (algebraically) freely adding type theoretic structure.

# Example #1: set of basic types

We'll construct the free natural model on the theory with an $I$-indexed family of basic types.

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# Example #2: freely admitting Σ-types

**Goal:** Given a natural model $(\mathbb{C}, p)$, construct the 'smallest' natural model $(\mathbb{C}_\Sigma, p_\Sigma)$ which extends $(\mathbb{C}, p)$ and admits Σ-types.

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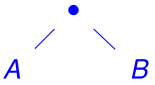

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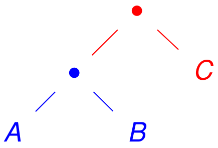

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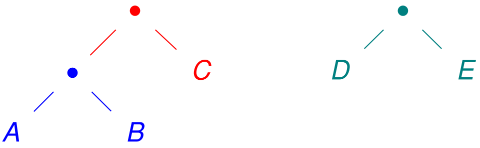

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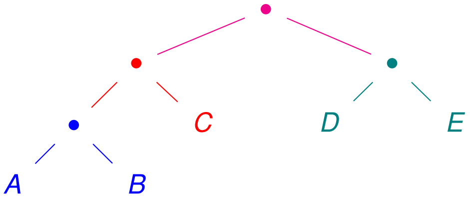

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There is a morphism $I : (\mathbb{C}, p) \to (\mathbb{C}_\Sigma, p_\Sigma)$, which sends types and terms to trivial trees (one vertex, no edges).

# Example #2: freely admitting Σ-types

**Theorem**

$(\mathbb{C}_\Sigma, p_\Sigma)$ *is a natural model admitting* $\Sigma$*-types*

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## Theorem

*$(\mathbb{C}_\Sigma, p_\Sigma)$ is a natural model admitting Σ-types, and for all $F : (\mathbb{C}, p) \to (\mathbb{D}, q)$ with $(\mathbb{D}, q)$ admitting Σ-types,*

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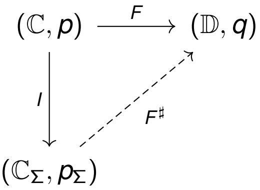
*$(\mathbb{C}_\Sigma, p_\Sigma)$ is a natural model admitting Σ-types, and for all $F : (\mathbb{C}, p) \to (\mathbb{D}, q)$ with $(\mathbb{D}, q)$ admitting Σ-types, there is a unique Σ-type-preserving $F^\sharp : (\mathbb{C}_\Sigma, p_\Sigma) \to (\mathbb{D}, q)$ extending $F$ along $I$.*



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**Inspiration:** Given a set $S$, the set of finite rooted binary trees with leaves labelled by elements of $S$ is an initial algebra for the polynomial functor $X \mapsto S + X \times X$.

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## Theorem

*$p_\Sigma$ is an initial algebra for the endofunctor $f \mapsto p + f \cdot f$.*

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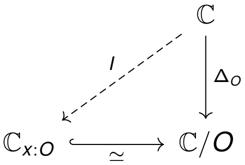

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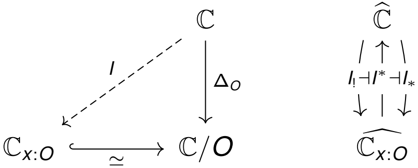

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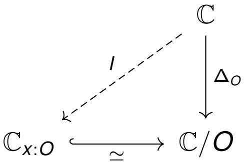


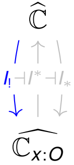

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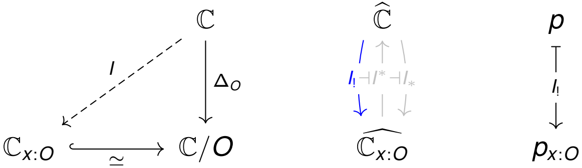

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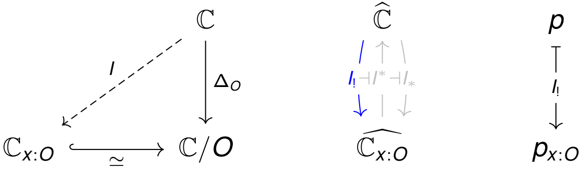
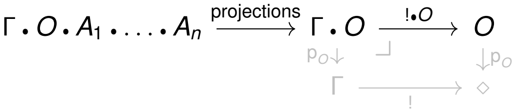



**Note:** The objects of $\mathbb{C}_{x:O}$ look like



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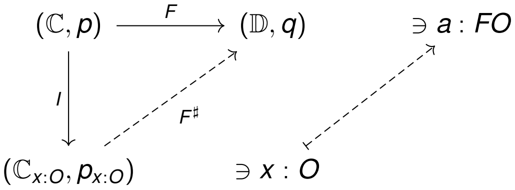

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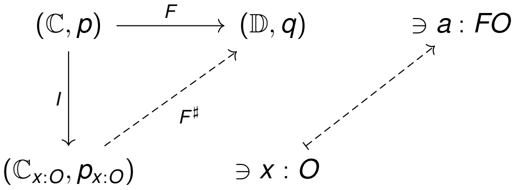


**Note:** $x$ is given by the diagonal map $O \to O \times O$ $(= \diamond \bullet O \bullet O[\mathsf{p}_O])$ in $\mathbb{C}/O$.

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- Formalise natural models in HoTT.

# Thanks for listening!

# References


## Natural models

- Awodey (2016) *Natural models of dependent type theory*, `arXiv:1406.3219`
- Awodey & Newstead (2018) *Polynomial pseudomonads and dependent type theory*, `arXiv:1802.00997`
- Fiore (2012) *Discrete generalised polynomial functors*, slides from talk at ICALP 2012

## Polynomials

- Gambino & Kock (2009) *Polynomial functors and polynomial monads*, `arXiv:0906.4931`

## Related work with CwFs

- Clairambault & Dybjer (2011) *The biequivalence of locally cartesian closed categories and Martin-Löf type theories*, `arXiv:1112.3456`