Cartesian Cubical Computational Type Theory

Favonia

U of Minnesota

Joint work with Carlo Angiuli, Evan Cavallo, Daniel R. Grayson, Robert Harper, Anders Mörtberg and Jonathan Sterling

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Some History

Coquand's notes 20??
BL 2014

AHW 2016 (CHTT Part I)
Cartesian cubical + computational

AH 2017 (CHTT Part II)
Dependent types

AFH 2017 (CHTT Part III)
Univalent Kan universes

CH 2018 (CHTT Part IV)
Higher inductive types
Identification types

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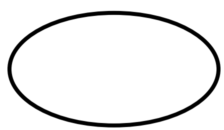
New Features of HoTT

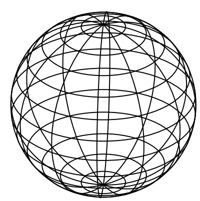
Univalence

if e is an equivalence between
types A and B, then ua(E):A=B

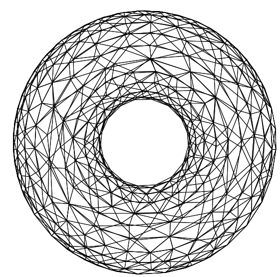
Higher Inductive Types



circle



sphere



torus

Equality and Paths

Definitional Equality

Silent in theory

If $A \equiv B$ and $M : A$ then $M : B$

Paths

Visible in theory

If $P : \mathrm{Path}(A, B)$ and $M : A$ then $\mathrm{transport}(M,P) : B$

Not Math Equality!

Definitional Equality Issue #1

Not very extensional

$$x : \mathbb{N}, y : \mathbb{N} \vdash x + y \not\equiv y + x : \mathbb{N}$$

(various reasonable trade-offs)

Not Math Equality!

Definitional Equality Issue #2

$$\text{winding} : \pi_1(S^1) \longrightarrow \mathbb{Z}$$

$$\text{winding}(\text{loop}) \not\equiv \text{any numeral}$$

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Canonicity

For any $M : \mathbb{N}$, there is a numeral N^* such that $\vdash M \equiv N^* : \mathbb{N}$

Restore Canonicity

Canonicity for $\mathbb{N}$ means canonicity for *every* type

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$$M : \mathbb{N} \times A$$
$$\mathrm{fst}(M) \equiv ??? : \mathbb{N}$$

Want $M \equiv \langle P, Q \rangle$ and then
$\mathrm{fst}(M) \equiv \mathrm{fst}\,\langle P, Q \rangle \equiv P \equiv$ some numeral

Restore Canonicity

But canonicity fails for paths!

$$\frac{M : A}{\mathrm{refl}(M) : M =_A M} \qquad \frac{a{:}A \vdash R : C(a,a,\mathrm{refl}(a)) \qquad P : M =_A N}{J(a.R,\ P) : C(M,N,P)}$$

$$\frac{a{:}A \vdash R : C(a,a,\mathrm{refl}(a)) \qquad M : A}{J(a.R,\ \mathrm{refl}(M)) \equiv R[M/a] : C(M,M,\mathrm{refl}(M))}$$

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$$J(\mathsf{ua}(E),\ x.N) \equiv \,???$$

$$J(\mathsf{loop},\ x.N) \equiv \,???$$

Restore Canonicity

Can we have canonicity + univalence?

Yes with De Morgan cubes [CCHM 2016]
Yes with Cartesian cubes [AFH 2017]

And higher inductive types?

Important examples with De Morgan cubes [CHM 2018]
Yes with Cartesian cubes [CH 2018]

Cubes

Idea: each type manages its own paths

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$$\text{loop} : \text{base} = \text{base}$$

Cubes

Idea: each type manages its own paths

~~loop : base = base~~

loop_x: a constructor of *the circle*

$$x{:}\mathbb{I} \vdash \mathrm{loop}_x : \mathrm{S1}$$

$$\mathrm{loop}_0 \equiv \mathrm{base} : \mathrm{S1} \qquad \mathrm{loop}_1 \equiv \mathrm{base} : \mathrm{S1}$$

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0{:}\mathbb{I} \qquad \Gamma \vdash 1{:}\mathbb{I}$$

$$\Gamma, x{:}\mathbb{I}, \Gamma' \vdash x{:}\mathbb{I}$$

$$x_1{:}\mathbb{I}, x_2{:}\mathbb{I}, \ldots, x_n{:}\mathbb{I} \vdash M : A$$

⇔ M is an n-cube in A

Cartesian Cubes

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$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

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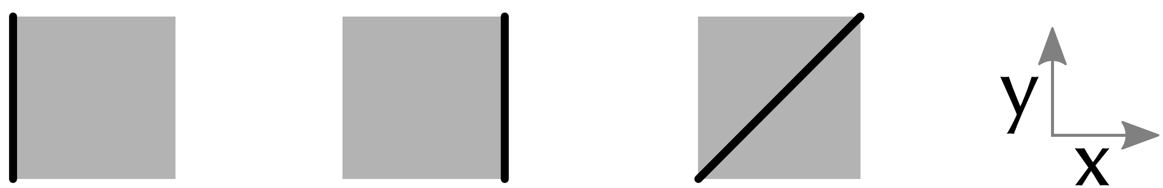
Cartesian: works as normal contexts

$M\langle 0/x \rangle$ $\quad$ $M\langle 1/x \rangle$ $\quad$ $M\langle y/x \rangle$



Ordinary Types

Ordinary typing rules hold uniformly

$$\frac{\Gamma,\ a{:}A \vdash M : B}{\Gamma \vdash \lambda a.M : (a{:}A) \longrightarrow B}$$

with any number of $\mathbb{I}$ in the Γ

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$$F(M_x\langle 0/x\rangle) \xrightarrow{\ \cancel{ap_F(M)}\quad F(M_x)\ } F(M_x\langle 1/x\rangle)$$

New Path Types

Dimension abstraction

$$\frac{x{:}\mathbb{I} \vdash M : A}{\langle x\rangle M : \mathrm{Path}_{x.A}(M\langle 0/x\rangle, M\langle 1/x\rangle)}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@r : A\langle r/x\rangle}$$

$$\frac{x{:}\mathbb{I} \vdash M : A}{(\langle x\rangle M)@r \equiv M\langle r/x\rangle : A\langle r/x\rangle}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@0 \equiv N_0 : A\langle 0/x\rangle}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@1 \equiv N_1 : A\langle 1/x\rangle}$$

Kan 1/2: Coercion

$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$

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$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$

M $\qquad$ $\mathsf{coe}_{x.A}[0 \rightsquigarrow x](M)$ $\qquad$ $\mathsf{coe}_{x.A}[0 \rightsquigarrow 1](M)$

$A\langle 0/x\rangle$ $\longrightarrow$ $A\langle 1/x\rangle$

$$\mathsf{coe}_{x.A}[r \rightsquigarrow r](M) \equiv M : A\langle r/x\rangle$$

Kan 2/2: Homogeneous Comp

$$\mathrm{hcom}_A[r \rightsquigarrow r'](M; \ldots, r_i{=}r'_i \hookrightarrow y.N, \ldots) : A$$

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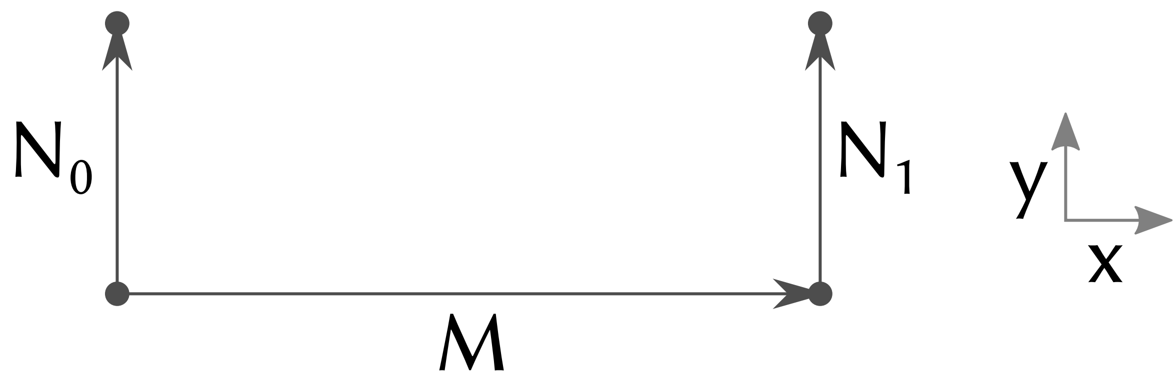
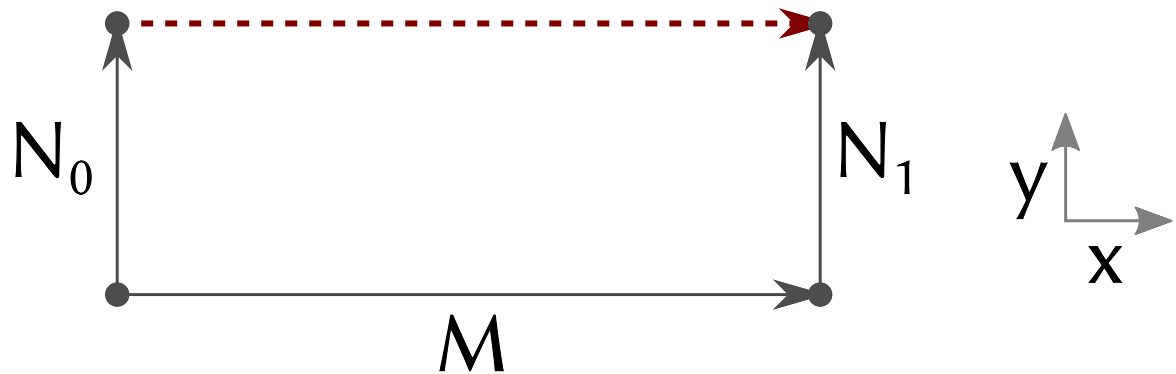


Kan 2/2: Homogeneous Comp

$\mathrm{hcom}_A[r \rightsquigarrow r'](M; \ldots, r_i = r'_i \hookrightarrow y.N, \ldots) : A$

$\mathrm{hcom}_A[0 \rightsquigarrow 1](M; x=0 \hookrightarrow y.N_0, x=1 \hookrightarrow y.N_1]$



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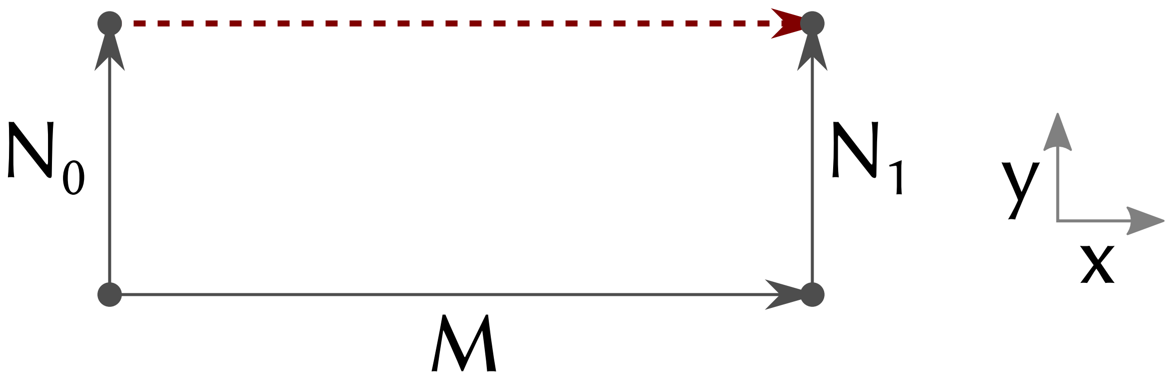
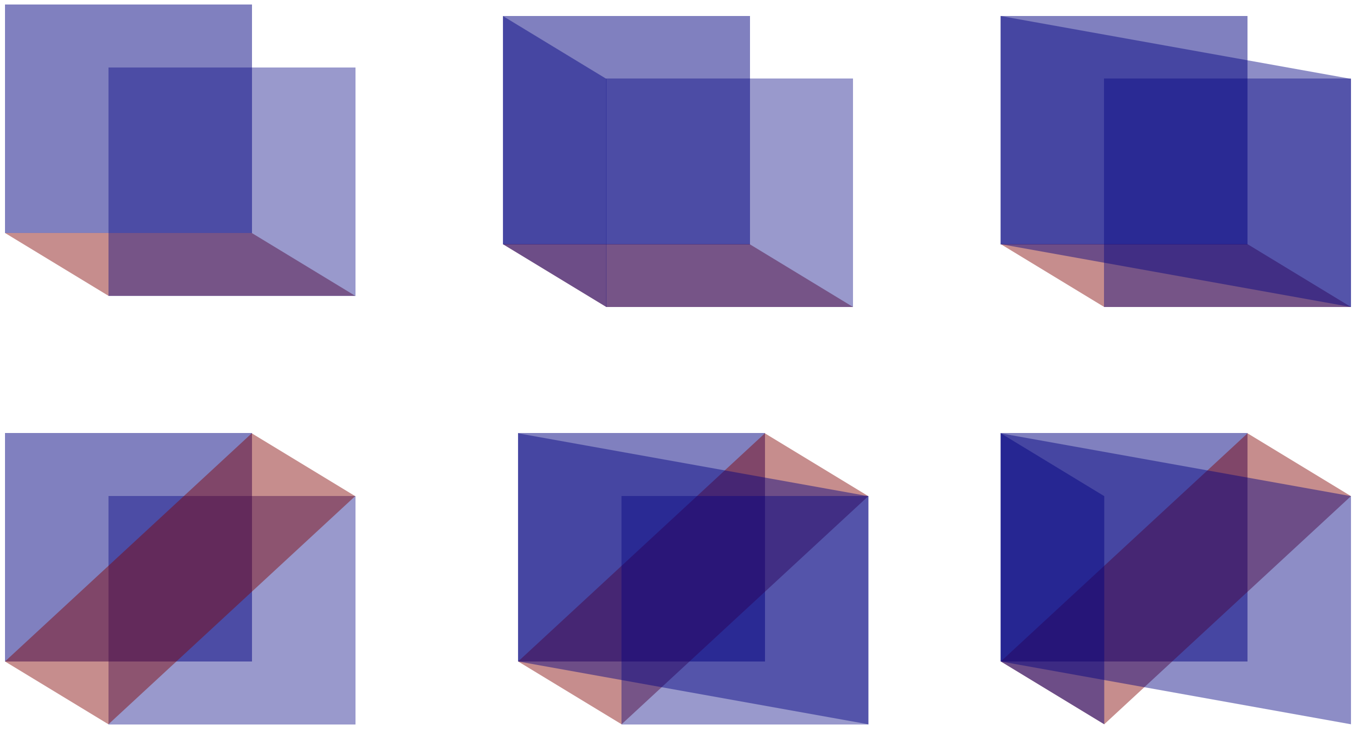


$\mathrm{hcom}_A[r \rightsquigarrow r](M; \ldots) \equiv M : A$

$\mathrm{hcom}_A[r \rightsquigarrow r'](M; \ldots, r_i{=}r_i \hookrightarrow y.N_i, \ldots) \equiv N_i\langle r'/y\rangle : A$

Kan 2/2: Homogeneous Comp



Fiberwise Fibrant Replacement

(the cubical way)

S1: hcom as the third constructor

Fiberwise Fibrant Replacement

(the cubical way)

S1: hcom as the third constructor

add only homogeneous ones
⇨ compat with base changes
⇨ no size blow-up!

(known by many experts in cubical TT)

Fiberwise Fibrant Replacement

(the cubical way)

If A has coercion,
the replacement of raw susp(A) is Kan

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(the cubical way)

If A has coercion,
the replacement of raw susp(A) is Kan

If objects on a span have coercion,
the replacement of raw pushout is Kan
(Note: the raw pushout might not have coercion!)

Important examples with De Morgan cubes [CHM 2018]
A general schema with Cartesian cubes [CH 2018]

Univalent Universes

$V_x(A,B,E)$ type

A line between $A\langle 0/x\rangle$ and $B\langle 1/x\rangle$
witnessed by the equivalence E

$$\frac{A \text{ type } [r=0] \qquad B \text{ type} \qquad E : A \simeq B\ [r=0]}{V_r(A,B,E) \text{ type}}$$

$$\frac{}{V_0(A,B,E) \equiv A} \qquad \frac{}{V_1(A,B,E) \equiv B}$$

expert only

Univalent *Kan* Universes

$\mathrm{hcom}_{U}[r \rightsquigarrow r'](A; \ldots)$ type

Make the universes Kan

Major difficulty:

Kan structure of the hcom types themselves

(Good news: greatly simplified after Part III is out)

Oh, Diagonals!

$$\mathrm{coe}_{x.\mathrm{hcom}[s \rightsquigarrow s'](A;\, \ldots)}[r \rightsquigarrow r'](M)$$

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$$\mathrm{coe}_{x.\mathrm{hcom}[s \rightsquigarrow s'](A;\ \ldots)}[r \rightsquigarrow r'](M)$$

when $s=s' \longmapsto \mathrm{coe}_{x.A}[r \rightsquigarrow r'](M)$

when $r=r' \longmapsto M$

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when $r=r' \longmapsto M$

$$\mathrm{hcom}[s \rightsquigarrow s'](\ldots,\ r=r' \hookrightarrow \ \ldots)$$

diagonals for coherence conditions

Computational Semantics

Transition system for closed terms

$$\lambda a.M \ \mathsf{val} \qquad (\lambda a.M)N \mapsto M[N/a]$$

$$\langle x \rangle M \ \mathsf{val} \qquad (\langle x \rangle M)@r \mapsto M\langle r/x \rangle$$

Computational Semantics

Transition system for closed terms

(other than dim. vars)

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Computational semantics: values

Canonicity as a corollary

Computational Semantics

Directly usable as a type theory

$$x : \mathbb{N}, y : \mathbb{N} \gg x + y \doteq y + x \in \mathbb{N}$$

with all the extensionalities

See our Part III for details

Implementations

RedPRL

In Nuprl style

redprl.org

redtt

(work in progress)

github.com/RedPRL/redtt

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yacctt

A proof of concept based on cubicaltt

github.com/mortberg/yacctt

Open Problems for HoTT

	Cubical	Simplicial
Standard?	???	Yes
HITs?	Yes	???

Open Problems for HoTT

	Cubical	Simplicial
Standard?	???	Yes
HITs?	Yes	???

HoTTopia

Very general construction with HITs

Summary of Cartesian Cubes

We have everything!

Univalent Kan universes

Higher inductive types

Identification types

...and proof assistants

redprl.org

github.com/RedPRL/redtt

github.com/mortberg/yacctt