Injective types in univalent mathematics

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Originally advertised contents

Interactions between

1. Injective types.
2. Yoneda Lemma (for types seen as ∞-groupoids).
3. Searchable types.
4. Ordinals (well-founded, transitive, extensional orders).
5. Topology and constructive mathematics.

The actual contents will be (1) and (2) only, perhaps with some verbal side remarks about the other things during the talk, not recorded in these slides.

(But recorded in Agda code.)

Injective objects

An object A is called injective over $e : X \hookrightarrow Y$ if every $f : X \to A$ extends to some $\bar{f} : Y \to A$ along e:

$$\begin{tikzcd} X \arrow[rr, hook, "e"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "\bar{f}"] \\ & A & \end{tikzcd}$$

Counter-example in a category of continuous maps of spaces:

$$\begin{tikzcd} \mathbb{R} \setminus \{0\} \arrow[rr, hook] \arrow[dr, "1/x"'] & & \mathbb{R} \arrow[dl, dotted, "\text{nothing here}"] \\ & \mathbb{R} & \end{tikzcd}$$

Hence $\mathbb{R}$ is not injective.

A couple of examples

1. Dana Scott (1972, LNM 274). TFAE for a T_0 topological space A:
   - A is injective over subspace embeddings.
   - A is a retract of a cartesian power of the Sierpinski space.
   - A is the set of points of a continuous lattice equipped with the Scott topology.

   Moreover, in this case, every $X \to A$ has a largest extension $Y \to A$ along any embedding $X \hookrightarrow Y$ in the topological specialization order, which coincides with the lattice order.

2. TFAE in an elementary topos:
   - A is injective over monomorphisms.
   - A is a retract of a power of the subobject classifier Ω.

What happens in univalent mathematics?

1. Equality moves from subsingleton-valued $X \times X \to \Omega$ to type-valued $X \times X \to \mathcal{U}$, becoming structure rather than property.
2. The natural notion of embedding $e : X \to Y$ is given by any of the following equivalent conditions:
   - The map $\mathsf{ap}\, e : x = x' \to e\, x = e\, x'$ is an equivalence for all $x, x' : X$.
   - The fibers $e^{-1}(y) := \Sigma(x : X), e\, x = y$ are all subsingletons.
   - The corestriction of e to its image $\Sigma(y : Y), \exists(x : X), e\, x = y$ is an equivalence.

   This implies that $e : X \to Y$ is left-cancellable,

   $$e\, x = e\, x' \to x = x'.$$

   - However, e can be left-cancellable in zero, one or more ways.
   - But it can be an embedding in at most one way.
   - If the domain X of e is a set, then left-cancellability becomes a proposition and hence coincides with the notion of embedding.

A notion of injective type

We will consider injectivity over such univalent embeddings.

1. Like in topological spaces and in toposes, extensions are not unique, but there are canonical extensions when there is any.
   - Here we have a smallest one and a largest one.
   - Or rather a left Kan extension and a right Kan extension.

   Hence it seems natural to use Σ rather than $\exists$ to formulate the extension property.
2. A type A is injective if for every embedding $e : X \to Y$ and every map $f : X \to A$ we can associate an extension $\bar{f} : Y \to A$ as in the diagram

$$\begin{tikzcd} X \arrow[rr, hook, "e"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "\bar{f}"] \\ & A & \end{tikzcd}$$

   That is, $\Pi(e : X \hookrightarrow Y), \Pi(f : X \to A), \Sigma(\bar{f} : Y \to A), \bar{f} \circ e = f$.
3. So injectivity is structure on A rather than property of A here.

Characterization of the injective types in univalent mathematics

- The injective types are precisely the retracts of exponential powers of $\mathcal{U}$.

  Moreover:

- The injective sets are precisely the retracts of powers of Ω.

- The injective 1-groupoids are precisely the retracts of powers of Set.

  $\vdots$

- The injective $n+1$-groupoids are precisely the retracts of powers of the type of n-groupoids.

I want to give the main steps of the proof/construction.

Step 1. $\mathcal{U}$ is injective

The "smallest" extension:

$$\begin{tikzcd}
X \arrow[rr, hook, "e"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "f \setminus e(y) :\equiv \Sigma(w : e^{-1}(y)), f(\mathrm{pr}_1\ w)"] \\
& \mathcal{U} &
\end{tikzcd}$$

- **Lemma.** A sum indexed by a subsingleton is equivalent to any of its summands.

  Corollary. For any $x : X$ we have $f \setminus e(e\,x) \simeq f(x)$ if e is an embedding, and hence $f \setminus e(e\,x) = f(x)$ by univalence.

- **Proposition.** If $y : Y$ is not in the image of e, then $f \setminus e(y) \simeq \mathbb{0}$.

- **Proposition.** $(\Sigma(x : X), f\ x) \simeq (\Sigma(y : Y), f \setminus e(y))$,

  without the need for excluded middle.

Step 1. $\mathcal{U}$ is injective

The "largest" extension:

$$\begin{tikzcd} X \arrow[rr, hook, "e"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "f/e(y) := \Pi(w : e^{-1}(y)), f(\mathrm{pr}_1\ w)"] \\ & \mathcal{U} & \end{tikzcd}$$

- **Lemma.** A product indexed by a subsingleton is equivalent to any of its factors.

  Corollary. For any $x : X$ we have $f/e(e\,x) \simeq f(x)$ if e is an embedding, and hence $f/e(e\,x) = f(x)$ by univalence.

- **Proposition.** If $y : Y$ is not in the image of e, then $f/e(y) \simeq \mathbb{1}$.

- **Proposition.** $(\Pi(x : X), f\,x) \simeq (\Pi(y : Y), f/e(y))$,

  again without the need for excluded middle.

  In our applications to searchable sets, we will instead consider $\Sigma(y : Y), f/e(y)$.

Application to the indiscreteness of the universe

(A sequential space is indiscrete iff every sequence converges to every point.)

$$\begin{tikzcd} \mathbb{N}+\mathbb{1} \arrow[rr, hook] \arrow[dr] & & \mathbb{N}_\infty \arrow[dl, dotted] \\ & \mathcal{U} & \end{tikzcd}$$

$\mathbb{N}_\infty$ = type of decreasing binary sequences.

1. We have the decreasing sequences $1^n 0^\omega$, which give the left component.
2. We have the decreasing sequence 1^ω, which gives the right component.
3. The embedding is an equivalence iff LPO (limited principle of omniscience) holds.
4. But it has empty complement.
5. It is an example of a searchable type.
6. One-point compactification of discrete $\mathbb{N}$ in Johnstone's topological topos.
7. "Generic convergent sequence".
8. Homotopy-final coalgebra of $(-)+\mathbb{1}$.

Natural transformations

- Given two type families $f, g : X \to \mathcal{U}$, the type of natural transformations is

$$\mathsf{Nat}\ f\ g :\equiv \Pi(x : X), f\ x \to g\ x$$

- The automatic naturality condition for $\eta : \mathsf{Nat}\ f\ g$ is that for all $x, y : X$ and $p : x = y$ we have $\eta\ y \circ \mathsf{ap}\ f\ p = \mathsf{ap}\ g\ p \circ \eta\ x$:

$$\begin{tikzcd}
f\ x \arrow[r, equal, "\mathsf{ap}\ f\ p"] \arrow[d, equal, "\eta\ x"'] & f\ y \arrow[d, equal, "\eta\ y"] \\
g\ x \arrow[r, equal, "\mathsf{ap}\ g\ p"'] & g\ y
\end{tikzcd}$$

Kan extensions

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, "f"'] & \arrow[d, phantom, "\dagger"] & Y \arrow[dl, "g"] \\ & \mathcal{U} & \end{tikzcd}$$

Theorem. For all $f : X \to \mathcal{U}$, and $e : X \to Y$ and $g : Y \to \mathcal{U}$, where e is not assumed to be an embedding, we have equivalences

1. $\mathrm{Nat}\,(f \setminus e)\, g \simeq \mathrm{Nat}\ f\,(g \circ e)$,
2. $\mathrm{Nat}\ g\,(f / e) \simeq \mathrm{Nat}\,(g \circ e)\, f$.

In particular, we have natural transformations

1. $\mathrm{Nat}\ f\,((f \setminus e) \circ e)$,
2. $\mathrm{Nat}\,((f / e) \circ e)\, f$,

which are equivalences if e is an embedding.

Step 2 - abstract nonsense

1. Retracts of injectives are injective.

$$\begin{tikzcd}
X \arrow[rr, hook, "e"] \arrow[dr, "f"] \arrow[ddr, "s \circ f"'] & & Y \arrow[dl, "f/e"'] \arrow[ddl, "(s \circ f)/e"] \\
& A' \arrow[d, hook, shift right, "s"'] & \\
& A \arrow[u, shift right, "r"'] &
\end{tikzcd}$$

2. Exponential powers of injectives are injective. Given $f : X \to (Z \to A)$ consider its transpose $Z \to (X \to A)$, and regard it as a family of maps parametrized by Z. Extend each one, to get a family $Z \to (Y \to A)$. Then back-transpose, to get $\bar{f} : Y \to (Z \to A)$, and check that this extends f.
3. Hence retracts of powers of $\mathcal{U}$ are injective.

Step 3 - The converse

1. Any injective type is a retract of every type it embeds into.

   Just extend the identity map.

$$\begin{tikzcd} A \arrow[rr, hook, "e"] \arrow[dr, "\mathrm{id}"'] & & Y \arrow[dl, dotted, "r"] \\ & A & \end{tikzcd}$$

2. Any type embeds into a power of $\mathcal{U}$.

   We will use the Yoneda Lemma for types to establish this.

3. Hence every injective type is a retract of a power of $\mathcal{U}$.

The Yoneda Lemma for types (Egbert Rijke 2012)

1. Regard the identity type former of a type X as a function $\mathsf{Id} : X \to (X \to \mathcal{U})$.

2. For any $A : X \to \mathcal{U}$ and $x : X$, the transformation

$$\begin{array}{rcl} \tau_x : A\,x & \to & \underbrace{\mathsf{Nat}\,(\mathsf{Id}\;x)\,A}_{\Pi(y\,:\,X),\, x = y \to A\,y} \\ a & \mapsto & \lambda(y : X)\,(p : x = y), \mathsf{transport}\,A\,p\,a \end{array}$$

   is a natural equivalence

$$Ax \simeq \mathsf{Nat}\,(\mathsf{Id}\;x)\,A.$$

Moreover (M.H.E. 2015)

TFAE for any $A : X \to \mathcal{U}$,

1. The total space ΣA is contractible.
2. For every $x : X$ and $a : A\,x$, the map $\tau_x(a) : \mathsf{Nat}\,(\mathsf{Id}\ x)\,A$ is a natural equivalence.

2'. For every $x : X$ and $a : A\,x$, the map $\tau_x(a) : \mathsf{Nat}\,(\mathsf{Id}\ x)\,A$ has a natural section.

3. For every $x : X$, any transformation $\mathsf{Nat}\,(\mathsf{Id}\ x)\,A$ is a natural equivalence.

3'. For every $x : X$, any transformation $\mathsf{Nat}\,(\mathsf{Id}\ x)\,A$ has a natural section.

Corollary. A universe $\mathcal{U}$ is univalent if and only if $\Pi(X : \mathcal{U}), \mathsf{isContr}(\Sigma(Y : \mathcal{U}), X \simeq Y)$.
Proof. Consider $A\,Y :\!= X \simeq Y$.

The Yoneda Embedding for types (M.H.E. 2015)

Theorem. The map $\mathsf{Id} : X \to (X \to \mathcal{U})$ is an embedding assuming univalence.

Proof. For $A : X \to \mathcal{U}$, the Id-fiber of A is $\Sigma(x : X), \mathsf{Id}\ x = A$.

If the pair (x, p) is in the fiber for $x : X$ and $p : \mathsf{Id}\ x = A$, then

$$\mathsf{ap}\ \Sigma\ p \ : \ \Sigma(\mathsf{Id}\ x) = \Sigma A,$$

and hence, being equal to a contractible type, the type ΣA is contractible.
Then

$$\begin{array}{rcll} A\,x & \simeq & \mathsf{Nat}\,(\mathsf{Id}\ x)\,A & \text{(Yoneda)} \\ & = & \Pi(y : X), \mathsf{Id}\ x\,y \to A\,y & \text{(definition)} \\ & = & \Pi(y : X), \mathsf{Id}\ x\,y \simeq A\,y & \text{(because } \Sigma A \text{ is contractible)} \\ & = & \Pi(y : X), \mathsf{Id}\ x\,y = A\,y & \text{(by univalence)} \\ & = & \mathsf{Id}\ x = A & \text{(by function extensionality).} \end{array}$$

Applying the function $\Sigma : (X \to \mathcal{U}) \to \mathcal{U}$ to both sides, we get
$\Sigma A \simeq \Sigma(x : X), \mathsf{Id}\ x = A$, and because the type ΣA is contractible, so is
$\Sigma(x : X), \mathsf{Id}\ x = A$ by transport, which shows that $\mathsf{Id} : X \to (X \to \mathcal{U})$ is an embedding.

This concludes the characterization of the injective types

Theorem/construction. The injective types are the retracts of powers of universes.

As for injective $n+1$-types, X is an $n+1$-type, by definition, if $\mathsf{Id} : X \to (X \to \mathcal{U})$ factors through the map $(X \to \mathcal{U}) \to (X \to \mathcal{U}^n)$, where $\mathcal{U}^n$ is the type of n-types, and $\mathcal{U}^n$ is injective because n-truncation exhibits it as a retract of $\mathcal{U}$.