A Yoneda lemma-formulation of the univalence axiom

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The univalent perspective on the foundations of mathematics, which was based on the homotopic interpretation of Martin-Löf's intensional type theory[1] (ITT) by Voevodsky in [15], Awodey and Warren in [1], and inspired by Hofmann and Streicher's groupoid interpretation of ITT in [6], reinforced the type-theoretic approach to the subject and brought mathematics and computer science even closer. Univalent type theory (UTT) is the extension of ITT with the univalence axiom.

Voevodsky's axiom of univalence (UA), the most important univalent concept, reflects the standard mathematical practice of identifying isomorphic objects. According to it, an equivalence between two types generates a proof of their equality. The converse i.e., the generation of an equivalence between two types from a proof of their equality, follows easily from Martin-Löf's J-rule, the induction principle that corresponds to the inductive definition of equality between the elements of a type as a new type.

It is well-known that proofs and computations in constructive mathematics rely heavily on the choice of representations. The univalence axiom allows one to identify equivalent definitions and hence makes program and data refinement possible. Originally, UA was motivated by the classical model of univalent foundations in the category of simplicial sets that was developed in [8]. The constructive interpretation of UA in cubical sets, found e.g., in [2], is a milestone in the development of univalent foundations. Based on this model, an extension of ITT is formed, which is called cubical type theory (CTT), where every type has a "cubical" structure[2].

The central role of UA in univalent foundations of mathematics and the current position of univalent foundations in logical studies raises naturally the following question[3].

How can one explain UA to a mathematician in familiar terms?

Since UA does not hold in the standard set-theoretic interpretation of type theory, there is no immediate set-theoretic answer to this question. In [14], section 5.8, an inductive form of UA, the *equivalence induction*, a principle very close to Martin-Löf's J-rule, is formulated. As in the case of the J-rule, to prove something about all proofs of equality, it suffices to prove it about the reflexivity proofs, in the case of equivalence induction, to prove something about all equivalences, it suffices to prove it about the identity maps. A major difference between the J-rule and the equivalence induction though, is that the computational rule of the latter involves propositional equality, while the computational rule of the former involves definitional equality.

In [13] Rijke gave a type-theoretic formulation of Yoneda lemma and constructed it from the J-rule and the function extensionality axiom. In [5] Escardó took Rijke's type-theoretic formulation of Yoneda lemma as primitive and constructed the J-rule from it so that its computation rule holds definitionally. In [12] we give a categorical formulation of the univalence axiom. The main results that we present are the following.

1. We give a Yoneda lemma-formulation (`sY-UA`) of Voevodsky's axiom of univalence (UA), providing a categorical interpretation to it. In contrast to Voevodsky's formulation of univalence, the computation rules of `sY-UA` hold definitionally.

[1]See e.g., [9], [10] and [11]. The canonicity property of ITT i.e., the fact that every closed term of the type of natural numbers is reduced to a numeral, makes ITT a programming language. As it is mentioned in [4], this is "a major compelling aspect of ITT compared to non-constructive foundations such as set theory".

[2]As shown in [7], CTT also satisfies the canonicity property.

[3]I would like to thank G. Jäger, whose question on the importance of UA outside Martin-Löf Type Theory motivated our work.

2. Based on the work of Escardó in [5], and applying Coquand's technique in [3] of reducing the J-rule to transport and the contractibility of singleton types, we construct from `sY-UA` the principle of equivalence induction with computational rule involving definitional equality.

3. We construct from `sY-UA` Voevodsky's original univalence axiom, where Escardó's formulation of UA (found in [2], p.15) is an intermediate corollary.

Following Rijke, the main idea behind this Yoneda lemma-formulation of UA is to view the universe $\mathcal{U}$ as a locally small category with $\mathrm{Hom}(A, B) \equiv A \simeq_{\mathcal{U}} B$, the successor universe $\mathcal{U}'$ to $\mathcal{U}$ as **Set**, and a type family $P : \mathcal{U} \to \mathcal{U}'$ as a contravariant functor from $\mathcal{U}$ to $\mathcal{U}'$.

Our Yoneda lemma-formulation of univalence supports the definitional approach to the computational rules associated to the judgements of type theory, and reinforces the "proximity" of UA to the J-rule also from the categorical point of view.

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