Algebraic models of dependent type theory

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There are many approaches to the categorical semantics of dependent type theory, with some notions being very closely tied to the syntax of type theory (e.g. Cartmell’s contextual categories [3] and Voevodsky’s $C$-systems [4]) and some more closely related to homotopy theory (e.g. Joyal’s tribes [6]).

This talk is an overview of my PhD thesis, advised by Steve Awodey, which explores the notion of a natural model of dependent type theory.

A natural model is an essentially algebraic object consisting of a map of presheaves $p : \hat{U} \to U$ over a small category $\mathcal{C}$ together with data witnessing representability of $p$. Thinking of $\mathcal{C}$ as the category of contexts and substitutions of dependent type theory, the map $p$ can be thought of as a context-indexed function sending terms-in-context to their unique type-in-context; context extension is then modelled by representability of $p$ ([1, §1], [4, Appendix]).

**Type constructors and polynomial monads.** The first contribution of the thesis is to relate the type constructors of dependent type theory with the algebraic notions of monads and their algebras via the machinery of polynomials and polynomial functors [5].

Necessary and sufficient conditions under which a natural model $(\mathcal{C}, p)$ admits a unit type $1$, dependent sum types $\sum_{x:A} B(x)$ and dependent product types $\prod_{x:A} B(x)$, can be succinctly expressed in terms of the existence of certain pullback squares in the category $\mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Set}]$ of presheaves over $\mathcal{C}$ [1, §2]. By considering maps of presheaves as polynomials in $\mathcal{E}$, we can express these pullback squares as cartesian morphisms of polynomials as follows:

$$\text{natural model supports... unit type dependent sums dependent products}
\iff \text{there exist cartesian... } \eta : 1 \Rightarrow p \quad \mu : p \cdot p \Rightarrow p \quad \alpha : p(p) \Rightarrow p$$

It is natural to ask whether $\eta$ and $\mu$ give rise to a monad structure on $p$, and whether $\alpha$ gives $p$ the structure of a $p$-algebra. Unfortunately, the answer is ‘not quite’, since this would require equations like $1 \times A = A$ to hold strictly, but they do not. Instead, we obtain the weaker notions of a pseudomonad and a pseudoalgebra.

Specifically, the bicategory $\text{Poly}^\text{cart}_\mathcal{E}$ can be equipped with the structure of a tricategory $\mathcal{P}\text{ol}\mathcal{Y}^\text{cart}_\mathcal{E}$ such that a natural model $(\mathcal{C}, p)$ admits a unit type and dependent sum types if and only if $p$ is a pseudomonad in $\mathcal{P}\text{ol}\mathcal{Y}^\text{cart}_\mathcal{E}$, and $(\mathcal{C}, p)$ additionally admits dependent product types if and only if $p$ is a $p$-psuedoalgebra. Proving this is the content of [2].

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**Categories of natural models.** Next, we provide a functorial description of morphisms of natural models, assembling natural models into a category $\text{NM}$. We prove that this is equivalent to the category $\text{Mod}(T)$ of models of an essentially algebraic theory. It is known (e.g. [1]) that the latter coincides with the category $\text{CwF}$ of categories with families, and hence:

(i) The category $\text{NM}$ of natural models is equivalent to the categories of categories with families; of categories with attributes; and of discrete comprehension categories.

(ii) Contextual categories are precisely those natural models whose contexts are generated by a finite set of basic types.

**Initial natural models.** The *initiality conjecture* asserts that the syntax of dependent type theory itself has the structure of a model, which is the initial object of the category of all models.

We prove partial results in this direction in the setting of natural models: we construct the free natural model $(\mathcal{C}_\sigma,p_\sigma)$ on some signatures $\sigma$ for dependent type theory and, for these signatures, prove that interpretations of $\sigma$ in a suitably structured natural model $(\mathcal{C},p)$ correspond naturally with homomorphisms $(\mathcal{C}_\sigma,p_\sigma) \to (\mathcal{C},p)$. We also describe how to freely equip an arbitrary natural model with additional type theoretic structure.

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