Dependent Right Adjoint Types

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There is recent interest in modal dependent type theories, e.g. guarded type theory [1], nominal type theory [8], clocked type theory [7], cohesive type theory, crisp type theory [9], and spatial type theory [5]. In this work we describe the model theory of Fitch style modal dependent type theories which allows as to treat modalities which are not necessarily (co)monadic. These include nominal type theories, guarded type theory and clocked type theory. The goal of this work is to isolate and study a common construction of these modal type theories. The construction centers around an adjoint pair of operators, where the left adjoint is an operation on contexts and the right adjoint is an operation on families. For example in nominal type theory [8] these are given by the rules

$$\frac{\Gamma \vdash \qquad n \notin \Gamma}{\Gamma, [n : \mathsf{N}]} \qquad \frac{\Gamma, [n : \mathsf{N}] \vdash \mathsf{A}}{\Gamma \vdash \text{И}[n : \mathsf{N}]\mathsf{A}}$$

One can then think of the model construction of these calculi as one of finding a dependent form of the adjunction

$$\frac{LA \to B}{A \to RB}$$

That is, when the function space is given by a Π-type.

We formulate the theory of these dependent adjunctions in the (aptly named) Categories with families with dependent right adjoints (CwDRA).

Definition 1 (CwDRA)**.** A CwDRA is a CwF [4] with the following added structure:
A functor L:

$$\frac{\Gamma \vdash}{\mathsf{L}\Gamma \vdash} \qquad \frac{\gamma : \Delta \to \Gamma}{\mathsf{L}\gamma : \mathsf{L}\Delta \to \mathsf{L}\Gamma} \qquad \mathsf{Lid} = \mathsf{id} \qquad \mathsf{L}(\gamma \circ \delta) = \mathsf{L}\gamma \circ \mathsf{L}\delta$$

An operation on families:

$$\frac{\mathsf{L}\Gamma \vdash A}{\Gamma \vdash \mathsf{R}_\Gamma A} \qquad (\mathsf{R}_\Gamma A)[\gamma] = \mathsf{R}_\Delta(A[\mathsf{L}\gamma])$$

An invertible transpose operation on elements of families:

$$\frac{\mathsf{L}\Gamma \vdash a : A}{\Gamma \vdash \overline{a} : \mathsf{R}_\Gamma A} \qquad \frac{\Gamma \vdash b : \mathsf{R}_\Gamma A}{\mathsf{L}\Gamma \vdash \overline{b} : A} \qquad \overline{a[\mathsf{L}\gamma]} = \overline{a}[\gamma] \qquad \overline{\overline{a}} = a$$

Although not provided here, we also describe a syntax and term model of CwDRA which can be seen as a dependent version of the ones in [3].

In the model construction of these type theories it is often the case that one starts with adjoint pair of endofunctors on the category of contexts. It is then natural to ask, what are the sufficient (resp. necessary) conditions under which this adjunction lifts to a dependent adjunction? We give the answer to the first question "sufficient conditions" by another construction called CwF+A. Where the 'A' stands for adjunction.

Definition 2 (CwF+A)**.** A CwF+A is a CwF with an adjuntion $\mathsf{L} \dashv \mathsf{R}$ on contexts along with a lifting of the right adjoint to families, i.e.

$$\frac{\Gamma \vdash A}{\mathsf{R}\Gamma \vdash \mathsf{R}A} \qquad \frac{\Gamma \vdash t : A}{\mathsf{R}\Gamma \vdash \mathsf{R}t : \mathsf{R}A} \qquad \mathsf{R}A[\mathsf{R}\gamma] = \mathsf{R}(A[\gamma]) \qquad \mathsf{R}t[\mathsf{R}\gamma] = \mathsf{R}(t[\gamma])$$

Along with an isomorphism $\nu_{\Gamma,A} : \mathsf{R}\Gamma.\mathsf{R}A \to \mathsf{R}(\Gamma.A)$

$$\mathsf{Rp}_A \circ \nu_{\Gamma,A} = \mathsf{p}_{\mathsf{R}A} \qquad (\mathsf{Rq}_A)[\nu_{\Gamma,A}] = \mathsf{q}_{\mathsf{R}A} \qquad \nu_{\Gamma,A} \circ \langle \mathsf{R}\gamma, \mathsf{R}a \rangle = \mathsf{R}\langle \gamma, a \rangle$$

We call such R a weak CwF morphism.

Lemma 1. *A CwF+A is a CwDRA and the two notions are equivalent if the CwF is democratic (i.e. where every context comes from a type[2]).*

Moreover we provide a very general way to construct CwDRAs:

Theorem 1. *If $\mathscr{C}$ is a lex category and $L \dashv R$ are adjoint endofunctors on $\mathscr{C}$, then the Giraud construction [6] has the structure of CwDRA.*

An important class of examples is given by direct images of geometric endomorphisms of presheaf toposes. We will explain how this construction extends to a type theory with (univalent) universes. This will allow us to capture extensions of cubical type theory, such as guarded cubical type theory [1], in a natural way.

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