Robust Notions of Contextual Fibrancy

Andreas Nuyts
imec-DistriNet, KU Leuven, Belgium

May 14, 2018

We address two problems. First, in existing cubical models of HoTT [BCH14, Hub15, CCHM16], fibrancy of the dependent function type requires fibrancy of both domain and codomain. This is in contrast to what we see in category theory, where a functor space is already a category if the codomain is a category; the domain can be merely a quiver.

Secondly, in a two-level type system [Voe13, ACK16] that distinguishes between fibrant and potentially non-fibrant types, it is often not possible to provide an internal fibrant replacement operation as it does not commute with substitution.

Assume in a category with families [Dyb96] a class of morphisms called horn inclusions (despite the name, they are not required to be monomorphisms). We say that a morphism is fibrant if it is equipped with an operation that lifts all horn fillers (i.e. we consider proof-relevant, algebraic fibrancy). We say that a type T is fibrant if its weakening morphism $\pi : (\Gamma, x : T) \to \Gamma$ is fibrant (fig. 1). We may additionally designate some squares as morphisms between horn inclusions and require fibrant morphisms to lift horn inclusions naturally. We call a thusly defined notion of fibrancy **robust** if the class of horn inclusions that generates it, is closed under pullbacks, i.e. if for any horn inclusion $\eta : \Lambda \to \Delta$ and any morphism $\sigma : \Delta' \to \Delta$, the pullback $\eta' : \Lambda \times_\Delta \Delta' \to \Delta'$ is also a horn inclusion (and the pullback square is a morphism of horn inclusions). For robust notions of fibrancy, we can prove (1) that the dependent function type is fibrant if its codomain is, and (2) that if a fibrant replacement operation exists, then it commutes with substitution up to isomorphism. Proofs of a more general result are sketched at the end of the abstract.

Unfortunately, some interesting notions of fibrancy are non-robust. For example, in cubical homotopy type theory [BCH14, Hub15, CCHM16], we have a horn inclusion $(i : \mathbb{I}, i = 0) \to (i : \mathbb{I})$. The fact that fibrant types lift this horn inclusion, yields transport between path-connected types. However, if we take the pullback with respect to the map $(i : \mathbb{I}, i = 0 \vee i = 1) \to (i : \mathbb{I})$, then we obtain a morphism $(i : \mathbb{I}, i = 0) \to (i : \mathbb{I}, i = 0 \vee i = 1)$. If this morphism were also a horn inclusion and hence lifted by all fibrant types, then we would be able to *teleport* data between arbitrary types, which is clearly inconsistent.

We can remedy this by considering fibrancy in a context. Instead of considering horn inclusions $\eta : \Lambda \to \Delta$, we consider **damped horn inclusions** which are diagrams of the form

$$\Lambda \xrightarrow{\eta} \Delta \xrightarrow{\zeta} \Psi, \tag{1}$$

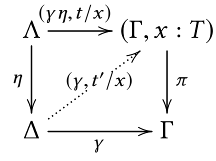

Figure 1: The sides of this diagram give a *horn* $(\gamma\eta, t/x)$ in $(\Gamma, x : T)$ which projects to a horn $\gamma\eta$ in Γ that extends over the horn inclusion η to a *filler* γ. As T is fibrant, this lifts to a filler $(\gamma, t'/x)$ of the horn $(\gamma\eta, t/x)$.

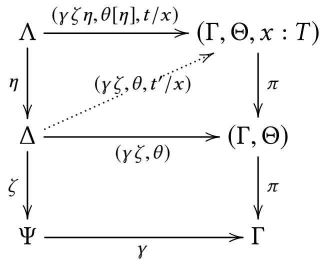

Figure 2: The sides of this diagram give a horn $(\gamma\zeta\eta, \theta[\eta], t/x)$ whose projection $(\gamma\zeta\eta, \theta[\eta])$ extends over the horn inclusion η to a filler $(\gamma\zeta, \theta)$. Because the further projection $\gamma\zeta$ of this filler factors over the *damping* ζ and because T is fibrant over Θ, the filler lifts to T.

and we say that a type $\Gamma; \Theta \vdash T$ fib (where Γ is a context and Θ is a dependent telescope over Γ) is fibrant over Θ in context Γ if it is equipped with an operation that, for every lifting problem as in fig. 2, gives a solution $\Delta \to (\Gamma, \Theta, x : T)$. This generalizes the notion of contextual fibrancy for cubical HoTT by Boulier and Tabareau [BT17]. It also subsumes the usual notion of fibrancy if we damp all horn inclusions with the identity damping: $\Lambda \to \Delta \to \Delta$. We call this notion of contextual fibrancy **robust** if it is closed under pullbacks, i.e. if for any damped horn inclusion $\Lambda \to \Delta \to \Psi$ and any $\sigma : \Psi' \to \Psi$, the diagram $\Lambda \times_\Psi \Psi' \to \Delta \times_\Psi \Psi' \to \Psi'$ is also a damped horn inclusion (and the pullback rectangle is a morphism of damped horn inclusions).

We can then (1) form fibrant dependent function types with non-fibrant domain, and (2) internalize the fibrant replacement operation if it exists. The rule F-Pi is natural in both Γ and Θ, whereas F-Repl is natural only in Γ and only up to isomorphism.

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma, x : A; \Theta \vdash T\,\mathsf{fib}}{\Gamma; \Theta \vdash (x : A) \to T\,\mathsf{fib}}\,\text{F-Pi} \qquad \frac{\Gamma, \Theta \vdash A\,\mathsf{type}}{\Gamma; \Theta \vdash \mathcal{R}_\Theta A\,\mathsf{fib}}\,\text{F-Repl} \tag{2}$$

Example 1. Boulier and Tabareau [BT17] pick as damped horn inclusions roughly the diagrams $(\Psi, i : \mathbb{I}, P \vee i = \epsilon) \to (\Psi, i : \mathbb{I}) \to \Psi$ for $\epsilon \in \{0, 1\}$, where Ψ is the Yoneda-embedding of some cube and P is a predicate over Ψ. If we remove the damping, then lifting these horn inclusions yields a box-filling operation that entails both composition and transport. What F-Pi (with damping) boils down to intuitively is that we can transport between function types, provided that the domain does not vary in the dimension i along which we transport; and that we can compose paths in the function type, provided that the domain varies over at most one of them.

Example 2. A category can be defined as a simplicial set that satisfies the Segal condition (due to Grothendieck) [Seg68], which states that any chain of n consecutive 1-simplices, can be extended to an n-simplex in a unique way. The 0-simplices then serve as objects, the 1-simplices as morphisms, and higher simplices as commutative diagrams. The underlying simplicial set is called the category's *nerve*. It is possible to define a class of horn inclusions in the category of simplicial sets, such that a simplicial set S is Segal if and only if the map $S \to \top$ to the terminal object, is algebraically fibrant. Indeed, existence of the filling n-simplex can be guaranteed by a horn inclusion $\Lambda_n \to \Delta_n$ where Λ_n is a chain of n consecutive 1-simplices and Δ_n is an n-simplex. Uniqueness is then guaranteed by a horn inclusion $\Delta_n \uplus_{\Lambda_n} \Delta_n \to \Delta_n$. Fibrant types $\Gamma \vdash T\,\mathsf{fib}$ over a (potentially non-Segal) simplicial set Γ then associate 0-simplices of Γ to categories, 1-simplices to profunctors, 2-simplices to profunctor morphisms $Q \circ P \to R$, and higher simplices to commutative diagrams. Clearly then, the type $(s : S) \to T[s]$ does not inherit this notion of fibrancy from S and $T[s]$, as contravariance breaks the action on 2-simplices.[1] Also, the fibrant replacement does not commute with substitution.

We can make this notion of fibrancy robust by damping the horn inclusions with arbitrary maps $\zeta : \Delta_n \to \Delta_1$ and then adding all pullbacks. Then, fibrant types $\Gamma; \Theta \vdash T\,\mathsf{fib}$ still associate 0- and 1-simplices to categories and profunctors. However, 2-simplices $(\gamma, \theta) : \Delta_2 \to (\Gamma, \Theta)$ are now associated to 'profunctor relations' between $Q \circ P$ and R and these need only be profunctor morphisms $Q \circ P \to R$ if γ is in fact degenerate on a 1-simplex.

The function type The idea behind the proof of F-Pi is that the lifting problem in fig. 2 for $T = ((x : A) \to B[x])$ is equivalent to a lifting problem for the damped horn inclusion $(\Lambda, x : A[\gamma\zeta\eta]) \to (\Delta, x : A[\gamma\zeta]) \to (\Psi, x : A[\gamma])$ in the type $\Gamma, x : A; \Theta \vdash B[x]\,\mathsf{fib}$. Now $B[x]$ is fibrant and the new damped horn inclusion is a pullback of the original one, so by robustness the lifting exists.

The fibrant replacement Pick a type $\Gamma, \Theta \vdash A\,\mathsf{type}$. Assume we can form a fibrant replacement $\Gamma; \Theta \vdash A'\,\mathsf{fib}$. We only show that this is natural in Γ with respect to *weakening* substitutions, i.e. we show that $\Gamma, z : C; \Theta \vdash A'\,\mathsf{fib}$ satisfies the universal property of the fibrant replacement. So pick another fibrant type $B[z]$ in the same context and a function $f[z] : A \to B[z]$; we have to show that $f[z]$ factors over $A \to A'$ as a structure-preserving function $f'[z] : A' \to B[z]$. Now since universal quantification is right adjoint to weakening, we may instead show that $\Gamma, \Theta \vdash \lambda x.\lambda z.f[z]\,x : A \to (z : C) \to B[z]$ factors over $A \to A'$, which is true as A' is the fibrant replacement of A (unweakened) and $(z : C) \to B[z]$ is fibrant. For general substitutions $\sigma : \Gamma' \to \Gamma$, we make the same argument using a generalization of the function type.

[1] If Γ is also Segal, then the projection $\pi : (\Gamma, s : S) \to \Gamma$ is a functor. The type $(s : S) \to T[s]$ is fibrant if and only if π is *Conduché*, meaning that it lifts factorizations in a suitable way [Gir64]. Fibrancy is then proven using similar techniques as in cubical HoTT [BCH14, Hub15, CCHM16].

Acknowledgements *The author holds a Ph.D. Fellowship from the Research Foundation - Flanders (FWO).*

References

[ACK16] Thorsten Altenkirch, Paolo Capriotti, and Nicolai Kraus. Extending homotopy type theory with strict equality. *CoRR*, abs/1604.03799, 2016. URL: `http://arxiv.org/abs/1604.03799`.

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: `http://drops.dagstuhl.de/opus/volltexte/2014/4628`, `doi:10.4230/LIPIcs.TYPES.2013.107`.

[BT17] Simon Boulier and Nicolas Tabareau. Model structure on the universe in a two level type theory. working paper or preprint, 2017. URL: `https://hal.archives-ouvertes.fr/hal-01579822`.

[CCHM16] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. *CoRR*, abs/1611.02108, 2016. URL: `http://arxiv.org/abs/1611.02108`.

[Dyb96] Peter Dybjer. *Internal type theory*, pages 120–134. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996. `doi:10.1007/3-540-61780-9_66`.

[Gir64] Jean Giraud. Méthode de la descente. *Bull. Soc. Math. Fr., Suppl., Mém.*, 2:115, 1964. `doi:10.24033/msmf.2`.

[Hub15] Simon Huber. A model of type theory in cubical sets. Licentiate's thesis, University of Gothenburg, Sweden, 2015.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Publications mathématiques de l'IHÉS*, 34(1):105–112, Jan 1968. URL: `https://doi.org/10.1007/BF02684591`, `doi:10.1007/BF02684591`.

[Voe13] Vladimir Voevodsky. A simple type system with two identity types. unpublished note, 2013. URL: `https://ncatlab.org/homotopytypetheory/files/HTS.pdf`.