The definitional symmetric cubical structure of types in type theory with equality defined by abstraction over an interval

Hugo Herbelin
INRIA - IRIF - University Paris Diderot

Abstract

Cohen, Coquand, Huber and Mörtberg [CCHM18] introduced a type theory whose equality type is defined as a (dependent) product over a formal notion of interval. This approach directly endows the tower of nested equalities over a type with a symmetric cubical structure whose equations over the operations of the structure hold definitionally.

We study a few properties of this structure from a typed perspective.

We consider a type theory with a universe U and a heterogeneous equality defined by dependent product over an interval ([CCHM18, Section 9]). We start with an interval with no particular structure, besides supporting variables i, j, k, ... and formal endpoints $\mathtt{0}$ and $\mathtt{1}$ (i.e. interval expressions are defined by $\tau ::= i \mid \mathtt{0} \mid \mathtt{1}$) . Typing contexts include declaration of interval variables. The rules for equality, essentially taken from [CCHM18], are the following ones:

$$\frac{\Gamma \vdash \xi : A =_{\widehat{U}} B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash t =_\xi u : \mathsf{U}} \qquad \frac{\Gamma \vdash p : t =_\xi u}{\Gamma \vdash p\,\mathtt{0} \equiv t} \qquad \frac{\Gamma \vdash p : t =_\xi u}{\Gamma \vdash p\,\mathtt{1} \equiv u}$$

$$\frac{\Gamma, i \vdash t : A}{\Gamma \vdash \lambda i.t : t[\mathtt{0}/i] =_{\lambda i.A} t[\mathtt{1}/i]} \qquad \frac{\Gamma \vdash v : t =_\xi u \qquad FV(\tau) \in \Gamma}{\Gamma \vdash v\,\tau : \xi\,\tau}$$

where we write $\widehat{t}$ to abbreviate $\lambda i.t$ for i not occurring in t (if t is of type A, $\widehat{t}$ is a proof of $t =_{\widehat{A}} t$ where $\widehat{A}$, this time with A of type U, is itself a proof of $A =_{\widehat{\mathsf{U}}} A$).

The type $A =_{\widehat{\mathsf{U}}} B$ can be seen as the type of lines connecting the types A and B. Let us call its inhabitants line types. For ξ a line type between A and B and for t of type A and u of type B, the type $t =_\xi u$ can be seen as the type of lines between t of type A and u of type B. An inhabitant of such a type is called a line and we say that it has line type ξ.

Let us then consider types A, B, C, and D, as well as lines ξ, ζ, ϕ and ψ relating these types as in the square drawn on the left below:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & C \\ {\scriptstyle\xi}\downarrow & \Rightarrow & \downarrow{\scriptstyle\zeta} \\ B & \xrightarrow[\psi]{} & D \end{array} \qquad\qquad \begin{array}{ccc} t & \xrightarrow{r} & v \\ {\scriptstyle p}\downarrow & \Rightarrow & \downarrow{\scriptstyle q} \\ u & \xrightarrow[s]{} & w \end{array}$$

This square can be identified with the type $\xi \underset{\phi \cong_{\widehat{\widehat{U}}} \psi}{=} \zeta$ where $\phi \cong_\xi \psi$ abbreviates $\lambda j.(\phi\, j =_{\xi\, j} \psi\, j)$. Its inhabitants we call square types.

Let us next consider t, u, v, and w of type A, B, C, and D respectively, and p, q, r and s lines between these points as drawn in the square above on the right, and E a square type, i.e. a proof of type $\xi \underset{\phi \cong_{\widehat{\widehat{U}}} \psi}{=} \zeta$ for some ξ, ζ, ϕ and ψ connecting A, B, C and D as in the figure. One can consider the type $p \underset{r \cong_E s}{=} q$, where $r \cong_E s$ again abbreviates $\lambda j.(r\, j =_{E\, j} s\, j)$. This can be seen as the type of squares with edges p, q, r and s and square type E.

We use the abbreviation $t \simeq^n_\xi u \triangleq \lambda i_1...i_n.(t\, i_1...i_n =_{\xi\, i_1...i_n} u\, i_1...i_n)$ and define 3-dimensional cube types as inhabitant of types of the form $E =_{G \simeq_{I \simeq^2_{\widehat{\widehat{\mathsf{U}}}} J} H} F$. An inhabitant of $\alpha =_{\gamma \simeq_{\eta \simeq^2_{\mathcal{E}} \theta} \delta} \beta$ is called a 3-dimensional cube, for α, β, γ, δ, η, θ squares with appropriate conditions on their boundaries, and $\mathcal{E}$ a cube type.

Calling points 0-cubes, lines 1-cubes and squares 2-cubes, we can more generally define n-cube types and n-cubes: given $2(n+1)$ n-cubes α_i and β_i for $0 \le i \le n$ and appropriate conditions on their boundaries, the previous nesting process allows to define a type of $(n+1)$-cubes over α_i and β_i and of n-cube type $\mathcal{E}$ that we shall abbreviate $[\alpha_0, ..., \alpha_n] =_{\mathcal{E}} [\beta_0, ..., \beta_n]$.

Let us now consider a general form of operations on (typed) n-cubes. An operation of dimension n to p is given by a triple (Φ, Ψ_0, Ψ_1) satisfying the following properties: (i) for any well-typed n-cube $\aleph$ of cube type $\mathcal{E}$ (i.e. $\aleph$ of some type $[\alpha_1, ..., \alpha_n] =_{\mathcal{E}} [\beta_1, ..., \beta_n]$), $\Psi_0(\aleph)$ and $\Psi_1(\aleph)$ are sequences of p faces such that $\Psi_0(\aleph) =_{\Phi(\mathcal{E})} \Psi_1(\aleph)$ is a well-typed type (ii) $\Phi(\aleph)$ is of this type (iii) $\Phi(t \simeq^n_\xi u) \equiv \Phi(t) \simeq^p_{\Phi(\xi)} \Phi(u)$ (together with similar rules for every other connective of the language). Examples of operations include:

- faces: $\partial^+(p) \triangleq p\,0$ and $\partial^-(p) \triangleq p\,1$, both of dimension 1 to 0 and both with $\Psi_0(p)$ and $\Psi_1(p)$ returning the empty list of faces;
- degeneracies/reflexivity: $\epsilon(t) \triangleq \widehat{t}$, of dimension 0 to 1 with both $\Psi_0(t)$ and $\Psi_1(t)$ returning the singleton list of faces $[t]$;
- transpositions/interchange: $\sigma(\alpha) \triangleq \lambda ij.\alpha\, j\, i$, of dimension 2 to 2, with $\Psi_0(\alpha) \triangleq [\lambda i.\alpha\, 0\, i, \lambda i.\alpha\, i\, 0]$ and $\Psi_1(\alpha) \triangleq [\lambda i.\alpha\, 1\, i, \lambda i.\alpha\, i\, 1]$;
- left (resp. right) connections: $\Gamma^+(p)$ (resp. $\Gamma^-(p)$) which can be taken as axioms, of dimension 1 to 2, with both $\Psi_0(p)$ and $\Psi_1(p)$ being $[p, \widehat{p\,1}]$ (resp. $[\widehat{p\,0}, p]$);
- reversals/inverses: p^{-1} which can be taken as an axiom, of dimension 1 to 1 with $\Psi_0(p) \triangleq p\,1$ and $\Psi_1(p) \triangleq p\,0$;
- diagonals: $\Delta(\alpha) \triangleq \lambda i.\alpha\, i\, i$, of dimension 2 to 1 with $\Psi_0(p) \triangleq \alpha\, 0\, 0$ and $\Psi_1(p) \triangleq \alpha\, 1\, 1$.

Operations from dimension 1 to some dimension p can be internalized as algebraic operations of arity p on the interval. For instance, reversal and connections can be obtained, as in [CCHM18], by extending the interval with $\tau ::= \ldots \mid -\tau \mid \tau \wedge \tau \mid \tau \vee \tau$ and defining $p^{-1} \triangleq \lambda i.p\,(-i)$, as well as $\Gamma^+(p) \triangleq \lambda ij.p\,(i \wedge j)$ and $\Gamma^-(p) \triangleq \lambda ij.p\,(i \vee j)$. Using iterated congruence, as defined by:

$$\widetilde{\Phi}^0(t) \triangleq \Phi(t) \qquad \widetilde{\Phi}^{m+1}(t) \triangleq \lambda i.\widetilde{\Phi}^m(t\, i) \qquad i \text{ taken fresh}$$

any operation Φ from dimension n can be extended into an operation Φ_m acting on cubes of dimension at least $m+n$. For instance, for $\aleph$ of dimension $q \ge 1$ and $0 \le m < q$, $\partial^+_m(\aleph) \triangleq \widetilde{\partial^+}^m(\aleph)$ is the m-th left face operation of the cubical structure.

Note in passing that any n-cube of type $[\alpha_1, ..., \alpha_n] =_{\mathcal{E}} [\beta_1, ..., \beta_n]$ can alternatively be seen as a p-cube of type $[\alpha_1, ..., \alpha_p] =_{[\alpha_{p+1}, ..., \alpha_n] \simeq^p_{\mathcal{E}} [\beta_{p+1}, ..., \beta_n]} [\beta_1, ..., \beta_p]$ for $0 \le p \le n$. Hence, any operation acting on a p-cube directly acts also on an $(p+q)$-cube.

Operations can also be extended to take several arguments, with composition or tensor product as examples.

References

[CCHM18] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom. In Tarmo Uustalu, editor, *TYPES 2015*, volume 69 of *LIPIcs*, pages 5:1–5:34. Schloss Dagstuhl, 2018.