(Truncated) Simplicial Models of Type Theory

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An important problem is the task of modeling higher category theory in (homotopy) type theory. In [5], Riehl and Shulman propose a *type theory with shapes* in which they develop a synthetic account to $(\infty, 1)$-*categories*, i.e. higher categories where all k-morphisms are invertible for $k > 1$. There are different ways of defining the notion of $(\infty, 1)$-category which however all agree up to homotopy.
In their type theory, Riehl and Shulman use an analogue of the notion of a *complete Segal space*, or *Rezk space* going back to Rezk [3]. A Rezk space is a kind of *simplicial space*, or more precisely, an object in the category of *bisimplicial sets* $\mathbf{bsSet} := \mathrm{PSh}_\Delta(\Delta) = [\Delta^{\mathrm{op}}, \mathbf{sSet}]$ that has homotopy coherent composition of all (higher) morphisms, and furthermore satisfies a kind of "univalence" condition. The category of bisimplicial sets with the Reedy model structure presents the $(\infty, 1)$-topos of simplicial ∞-groupoids, $\mathrm{PSh}_\infty(\Delta)$, and since Δ is an elegant Reedy category, we know from Shulman [7] that this structure provides a model of HoTT, i.e. intensional type theory with a univalent universe of small types.
However, the universes classifying the types with $(\infty, 1)$-categorical, or ∞-groupoidal structure are not themselves (complete) Segal types.
We discuss how this could be achieved in *truncated models* of simplicial type theory, living in $\mathrm{PSh}_\infty(\Delta_{\leq k})$ with $k = 1, 2$. In the case $k = 1$ this gives rise to a reflexive graph model of type theory (cf. Rijke–Spitters [6]).
Furthermore, an improvement of the base categories $A := \Delta_{\leq k}$ is given by their *direct replacements* $\mathcal{D}(A)$, a construction due to Szumiło [9] (also cf. Kraus–Sattler [2]). Now, $\mathcal{D}(A)^{\mathrm{op}}$ are *inverse* categories, and these allow for inductive definitions of type families. By work of Shulman [8], inverse categories give rise to well-behaved models of HoTT as well, and for finite inverse categories, these can even be described internally in type theory. Preservation of at least the geometric fragment of the theory is given by the geometric morphism $\mathrm{PSh}(\mathcal{D}(A)) \to \mathrm{PSh}(A)$ induced by the canonical "projection" map $\mathcal{D}(A) \to A$.
The simplicial type theories arising in these truncated models allow for defining synthetic $(\infty, 1)$-categories à la Riehl and Shulman but significantly simplify the investigations of universes. We present the ensuing type theories with universes, along with some

refinements having the same object types, but different arrow types, thus providing synthetic analogues of the $(\infty, 1)$-categories $\mathbf{Cat}_\infty$ or $\mathbf{Gpd}_\infty$ of $(\infty, 1)$-categories, or ∞-groupoids, resp.
A particularly intriguing application of this would be the synthetic theory of fibrations of $(\infty, 1)$-categories because it would lead to internal versions of the *Grothendieck constructions*, which classically say that cocartesian, or left fibrations of quasi-categories are classified by $\mathrm{Fun}_\infty(-, \mathbf{Cat}_\infty)$, or $\mathrm{Fun}_\infty(-, \mathbf{Gpd}_\infty)$, resp.

References

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