Differential Cohesive Type Theory [1]

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Differential Cohesive Toposes

$$\begin{array}{ccccc} \Re & \dashv & \Im & \dashv & \& \\ & & \cup & & \cup \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

$\Im$, $\int$ and $\sharp$ are reflections.
$\Re$, $\&$ and $\flat$ are coreflections.
$\int$ and $\Re$ preserve finite products.

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to get the site of *formal cartesian spaces*

$$\{\underbrace{\mathcal{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V)}_{=:(\mathbb{R}^n \times \mathbb{D}_V)^{\mathrm{op}}} \mid n \in \mathbb{N} \text{ and } V \text{ nilpotent, } \dim_{\mathbb{R}} V < \infty\}^{\mathrm{op}}$$

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On representables $\Re$ is given by reduction of $\mathbb{R}$-algebras:

$$\Re(\mathbb{R}^n \times \mathbb{D}_V) = \mathbb{R}^n$$