Interpreting Cubical Type Theory in Appropriate Presheaf Toposes

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Cubical Type Theory

- CTT: Extension of dependent type theory (with Σ, Π-types and a universe) by an **interval**, a **lattice of faces**, **path types** and certain operations for type families (**composition** and **glueing**).
- Devised by Bezem, Cohen, Coquand, Huber and Mörtberg [BCH14, CCHM16] in 2014-2016 as an intensional type theory which validates Voevodsky's **Univalence Axiom** and has **computational meaning**.
- Further developments by Orton and Pitts [OP16] as well as Birkedal, Bizjak, Clouston, Gratwohl, Spitters and Vezzosi [BBCGSV16] in 2016.
- **Goal of this talk:** Present semantics of CTT in $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ and $\mathcal{E}^{\mathcal{C}^{\mathrm{op}}}$, with $\mathcal{E}$ a model of extensional type theory (ETT) and $\mathcal{C}$ a category internal to $\mathcal{E}$.

Interval and face lattice I

The **interval** is a pretype $\mathbb{I}$ with constants $0, 1 : \mathbb{I}$ and operations

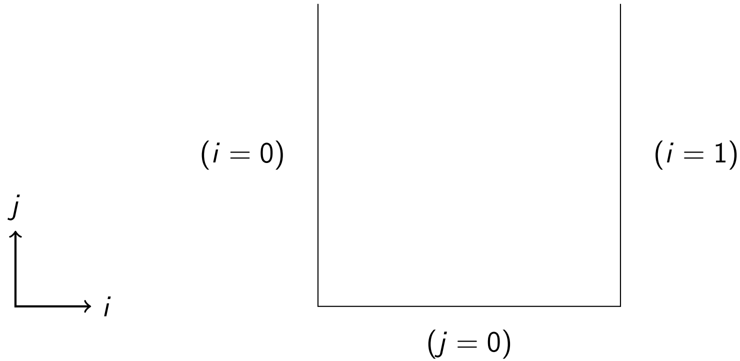
$$\sqcap, \sqcup : \mathbb{I} \to \mathbb{I} \to \mathbb{I},$$
$$1 - \cdot : \mathbb{I} \to \mathbb{I},$$

endowing $\mathbb{I}$ with the structure of a **de Morgan algebra** where 1 is indecomposable.

We obtain the **face lattice** $\mathbb{F}$ from $\mathbb{I}$ by factoring modulo $x \sqcap (1 - x) = 0$.

Interval and face lattice II

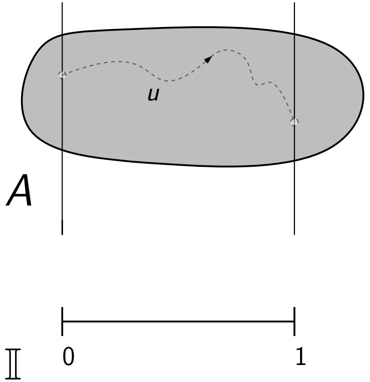
Example: $\varphi = (i = 0) \sqcup (i = 1) \sqcup (j = 0) : \mathbb{F}$



For $\Gamma \vdash \varphi : \mathbb{F}$, $\Delta \vdash \psi : \mathbb{F}$, ... we may form the **restricted contexts** (Γ, φ), (Δ, ψ), ...

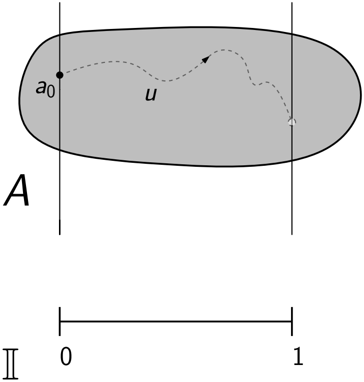
Composition operation

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash a_1 = \mathsf{comp}^i(A, \varphi, u, a_0) : A(i1)[\varphi \mapsto u(i1)]}$$



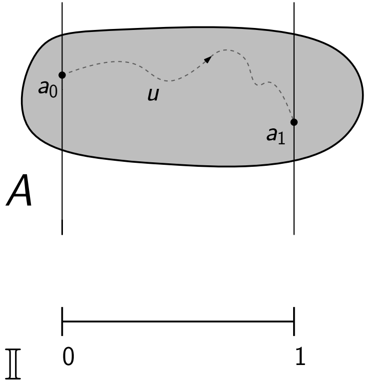
Composition operation

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash a_1 = \mathsf{comp}^i(A, \varphi, u, a_0) : A(i1)[\varphi \mapsto u(i1)]}$$



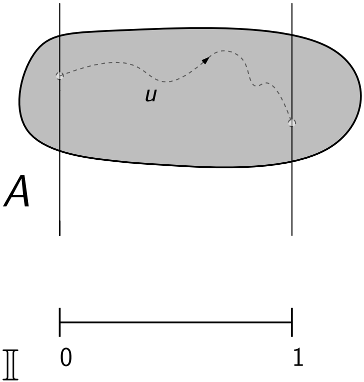
Composition operation

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash a_1 = \mathrm{comp}^i(A, \varphi, u, a_0) : A(i1)[\varphi \mapsto u(i1)]}$$



Composition operation

Composition is actually equivalent to **filling**:



Composition operation

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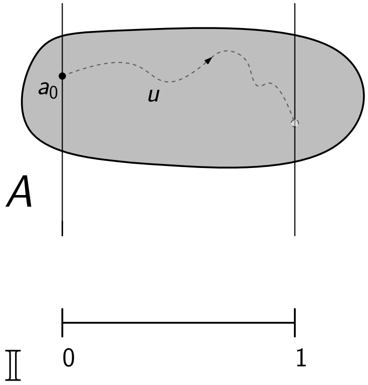
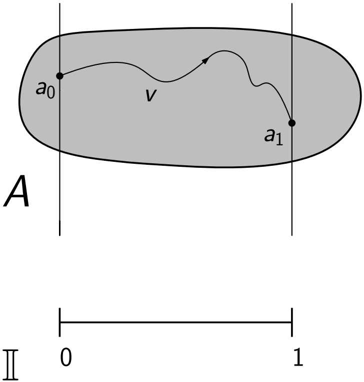


Composition operation

Composition is actually equivalent to **filling**:
$\Gamma, i : \mathbb{I} \vdash v = \mathrm{fill}^i(A, \varphi, u, a_0)$



Glueing operation I

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash w : T \to A}{\Gamma \vdash \mathsf{Glue}_\Gamma(\varphi, T, A, w)}$$

$$\frac{\Gamma \vdash b : \mathsf{Glue}_\Gamma(\varphi, T, A, w)}{\Gamma \vdash \mathsf{unglue}(b) : A[\varphi \mapsto w(b)]}$$

$$\frac{\Gamma, \varphi \vdash w : T \to A \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto w(t)]}{\Gamma \vdash \langle \mathsf{glue}(\varphi, t), a \rangle : \mathsf{Glue}_\Gamma(\varphi, T, A, w)}$$

s.t. **judgmentally**:

$$\begin{array}{ll} \mathsf{Glue}_\Gamma(1, T, A, w) = T & \mathsf{unglue}(\mathsf{glue}(\varphi, t), a) = t \\ \mathsf{glue}(1, T, A, w)(t) = t & \langle \mathsf{glue}(\varphi, t), \mathsf{unglue}(b) \rangle = b \end{array}$$

Glueing operation II

$$
\begin{tikzcd}
T \arrow[rr, dashed, "{\mathsf{glue}(\varphi,A,T,w)}"] \arrow[dr, "w"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddr, bend right] & & \mathsf{Glue}_\Gamma(\varphi, A, T, w) \arrow[dr, dashed, "{\mathsf{unglue}(\varphi,A,T,w)}"] \arrow[ddr, dashed] & \\
& \iota_\varphi^* A \arrow[rr, hook, "\iota_{\varphi,A}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & & A \arrow[d] \\
& (\Gamma, \varphi) \arrow[rr, hook, "\iota_\varphi"'] & & \Gamma
\end{tikzcd}
$$

s.t. **judgmentally**:

$$
\begin{aligned}
&\mathsf{Glue}_\Gamma(1, T, A, w) = T && \mathsf{unglue}(\mathsf{glue}(\varphi, t), a) = t \\
&\mathsf{glue}(1, T, A, w)(t) = t && \langle \mathsf{glue}(\varphi, t), \mathsf{unglue}(b)\rangle = b
\end{aligned}
$$

Cubical sets

Let $\mathbb{C}$ be a category such that:

1. $\mathbb{C}$ has finite products
2. $\widehat{\mathbb{C}}$ has an **interval object** $\mathbb{I}$ which is **representable** and **connected** in the sense that the canonical morphism $\mathbf{2} \to \mathbf{2}^{\mathbb{I}}$ is an isomorphism.
3. The weakening morphism $\mathbb{F} \to \mathbb{F}^{\mathbb{I}}$ has a right adjoint $\forall \colon \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$.

We call $\mathbb{C}$ "the" **category of cubes** and $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ "the" **category of cubical sets**.

Modelling CTT in presheaves I

Definition (Type category)

A **type category** consists of categories $\mathbb{B}$ and $\mathbb{E}$ with a discrete Grothendieck fibration $\mathcal{T} \colon \mathbb{E} \to \mathbb{B}$ and a cartesian functor $p \colon \mathbb{E} \to \mathbb{B}^{\to}$ from $\mathcal{T}$ to the codomain fibration:

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "p"] \arrow[dr, "\mathcal{T}"'] & & \mathbb{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbb{B} & \end{tikzcd}$$

Set $\mathbb{B} = \widehat{\mathbb{C}}$ and $\mathbb{E}_\Gamma = \widehat{\int_{\mathbb{C}} \Gamma} \simeq \widehat{\mathbb{C}}/\Gamma$ for $\Gamma \in \widehat{\mathbb{C}}$.

Modelling CTT in presheaves II

- Consider the classifying map $(\cdot = 1) \colon \mathbb{I} \to \Omega$ of the global element 1 of $\mathbb{I}$. Interpret the face lattice as $\mathbb{F} := \mathrm{im}(\cdot = 1)$:

$$\begin{tikzcd} \mathbb{I} \arrow[rr, "(\cdot=1)"] \arrow[dr] & & \Omega \\ & \mathbb{F} \arrow[ur, hook] & \end{tikzcd}$$

- Consider $\varphi \colon \Gamma \to \mathbb{F}$. Restricted contexts arise as pullbacks

$$\begin{tikzcd} \Gamma, \varphi \arrow[r] \arrow[d, hook, "\mathrm{Id}_{\mathbb{F}}(\varphi,1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{F} \arrow[d, tail, "\mathrm{Id}_{\mathbb{F}}"] \\ \Gamma \arrow[r, "\langle \varphi,1 \rangle"'] & \mathbb{F} \times \mathbb{F} \end{tikzcd}$$

and we write $[\varphi] := \mathrm{Id}_{\mathbb{F}}(\varphi, 1)$.

Universe of pretypes

Construct generic family $E \to U$ in $\widehat{\mathbb{C}}$ à la [HS97, Str05]:

- $\mathcal{U}$ Grothendieck universe hosting the category $\mathbb{C}$
- Define $U \in \widehat{\mathbb{C}}$ by

$$\begin{aligned} U(I) &:= \mathcal{U}^{(\mathbb{C}/I)^{\mathrm{op}}} \quad \text{for} \quad I \in \mathbb{C}, \\ U(u : J \to I)(A) &:= A \circ (\Sigma_u)^{\mathrm{op}} \quad \text{for} \quad u : J \to I. \end{aligned}$$

- Define E over U as the presheaf

$$\begin{aligned} E(\langle I, A \rangle) &:= A(\mathrm{id}_I), \\ E(u : \langle J, u^* A \rangle \to \langle I, A \rangle)(a) &:= A(u : u \to \mathrm{id}_I)(a). \end{aligned}$$

- **N.B.:** We get Ω when choosing $\mathcal{U} = \{0, 1\}$.

Fibrations

Definition (Composition structure (cf. [CCHM16], [OP16]))

Define the family $\mathsf{Comp} : \mathcal{T}(U^{\mathbb{I}})$ of **composition structures** as

$$\begin{aligned} \mathsf{Comp}(A : U^{\mathbb{I}}) \quad := \quad & (\Pi\varphi : \mathbb{F})(\Pi p : [\varphi] \to (\Pi i{:}\mathbb{I})A(i)) \\ & \{a \in A(0) \mid \forall u{:}[\varphi].p(u)(0) = a\} \\ & \to \{a \in A(1) \mid \forall u{:}[\varphi].p(u)(1) = a\}. \end{aligned}$$

Externally composition structures "put lids on open boxes".

Definition (Fibration structure (cf. [CCHM16], [OP16]))

Define the family $\mathsf{Fib} : \mathcal{T}(U^{\Gamma})$ of **fibration structures** as

$$\mathsf{Fib}_{\Gamma}(A) := (\Pi p{:}\Gamma^{\mathbb{I}})\mathsf{Comp}(A \circ p).$$

Universe of types

- Define $U_f \in \widehat{\mathbb{C}}$ by

$$U_f(I) := \{A = \langle |A|, \mathrm{fib}(A)\rangle \mid |A| \colon I \to U,\ \mathrm{fib}(A) : \mathrm{Fib}_I(|A|)\}.$$

- The **universe** $E_f \to U_f$ **of fibrant types** is obtained from $E \to U$ by pulling back along the forgetful map $U_f \to U$:

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ U_f \arrow[r] & U \end{tikzcd}$$

Interpreting glueing I

$$
\begin{tikzcd}
T \arrow[rr, dashed, "\mathsf{glue}(\varphi{,}A{,}T{,}w)"] \arrow[dr, "w"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddr, bend right] & & \mathsf{Glue}_\Gamma(\varphi, A, T, w) \arrow[dr, dashed, "\mathsf{unglue}(\varphi{,}A{,}T{,}w)"] \arrow[ddr, dashed] & \\
& \iota_\varphi^* A \arrow[rr, hook, "\iota_{\varphi,A}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & A \arrow[d] \\
& (\Gamma, \varphi) \arrow[rr, hook, "\iota_\varphi"'] & & \Gamma
\end{tikzcd}
$$

Strictness: $\mathsf{Glue}_\Gamma(1, A, T, w) = T$ and $\mathsf{glue}(1, A, T, w) = \mathrm{id}_T$

Interpreting glueing II

Strictness issue: difficult in general toposes, cf. [OP16]. But can be achieved in $\widehat{\mathbb{C}}$, cf. [CCHM16], as follows. Write $G := \mathrm{Glue}_\Gamma(\varphi, A, T, w)$ and for $I \in \mathbb{C}$, $\gamma \in \Gamma(I)$ let

$$G(I,\gamma) := \begin{cases} T(I, \langle \gamma, * \rangle) & \text{if } \varphi_I(\gamma) = 1 \\ \{\langle t, a\rangle \mid a \in A(I,\gamma), t \colon [\varphi]_I(\gamma) \to T, & \\ \qquad a_{|\varphi_I(\gamma)} = \iota_{\varphi,A} \circ w \circ t\} & \text{otherwise} \end{cases}$$

where $a_{|\varphi_I(\gamma)}$ is the restriction of a along the inclusion of $\varphi_I(\gamma)$ into $\mathbf{y}(I)$.

Interpreting glueing III

Reindexing: Let $u : J \to I$ in $\mathbb{C}$. Reindexing along $u : \langle J, u^*\gamma \rangle \to \langle I, \gamma \rangle$ is by case distinction:

$$
\begin{aligned}
\varphi_I(\gamma) = 1: &\qquad T(I, \langle \gamma, * \rangle) \ni t \longmapsto u^* t \in T(J, \langle u^*\gamma, * \rangle) \\
\varphi_I(\gamma) \neq 1: &\qquad \langle t, a \rangle \longmapsto \begin{cases} u^* a & \text{if } \varphi_J(u^*\gamma) = 1 \\ \langle u^* t, u^* a \rangle & \text{otherwise} \end{cases}
\end{aligned}
$$

Here $u^* a = a \circ \mathbf{y}(u)$, and $u^* t$ arises as in:

$$
\begin{tikzcd}
 & T \arrow[rr, "w"] & & \iota^* A \arrow[d, hook, "\iota_{\varphi, A}"] \\
u^*[\varphi]_I(\gamma) \arrow[ur, bend left, "u^* t"] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & {[\varphi]_I(\gamma)} \arrow[u, "t"] \arrow[rr, "a|_{\varphi_I(\gamma)}"] \arrow[d, hook] & & A \arrow[d] \\
\mathbf{y}(J) \arrow[r, "\mathbf{y}(u)"'] & \mathbf{y}(I) \arrow[urr, "a"] \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}
$$

Interpreting glueing IV

- The map $\mathrm{glue}(\varphi, A, T, w) \colon T \to G$ is defined by:

$$\mathrm{glue}(\varphi, A, T, w)_{I,\gamma}(t) := \begin{cases} t & \varphi_I(\gamma) = 1 \\ \langle t, w_{I,\gamma}(t) \rangle & \text{otherwise} \end{cases}$$

- The map

$$\mathrm{unglue}(\varphi, A, T, w) \colon G \to A$$

over Γ is given by:

$$\mathrm{unglue}(\varphi, A, T, w)_{I,\gamma}(b) := \begin{cases} w_{I,\gamma}(b) & \varphi_I(\gamma) = 1 \\ \mathrm{pr}_2(b) & \text{otherwise} \end{cases}$$

Composition for glueing I

For a type $p_A \colon A \to \Gamma$ let P_A be the **type of paths** in A interpreted as follows:

$$
\begin{tikzcd}
P_A = A^{\mathbb{I}} \arrow[rr, "\langle A^0, A^1 \rangle"] \arrow[dr, "(p_A)^{\Gamma^* \mathbb{I}}"'] & & A \times_\Gamma A \arrow[dl] \\
& \Gamma &
\end{tikzcd}
$$

Define **weak equivalences** á la Voevodsky:

- **Contractability:** $\mathrm{isContr}(A : U) := (\Sigma x : A)(\Pi y : A) P_A(x, y)$
- **Homotopy fiber:** $\mathrm{hfib}(A, B : U)(f : A \to B)(y : B) := (\Sigma x : A) P_B(f(x), y)$
- **Weak equivalences:** $\mathrm{isWeq}(A, B : U)(f : A \to B) := (\Pi y : B)\mathrm{isContr}(\mathrm{hfib}(A, B, f, y))$,
  $\mathrm{Weq}(A, B : U) := (\Sigma f : A \to B)\mathrm{isWeq}(A, B, f)$

Composition for glueing II

Theorem ([CCHM16], Sec. 6.2)

Let $\varphi \colon \Gamma \to \mathbb{F}$, $A \in \mathcal{T}(\Gamma \times \mathbb{I})$ *and* $T \in \mathcal{T}((\Gamma, \varphi) \times \mathbb{I})$ *have a composition structure, and* $w : \mathsf{Weq}(T, \iota_\varphi^* A)$. *Then* $G := \mathsf{Glue}_{\Gamma \times \mathbb{I}}(\varphi, A, T, w) \in \mathcal{T}(\Gamma \times \mathbb{I})$ *also has a composition structure.*

The proof makes crucial use of $\sqcap, \sqcup \colon \mathbb{I}^2 \to \mathbb{I}$ and the map $\forall \colon \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$, which is right adjoint to weakening.

Composition for the universe

Theorem ([CCHM16], Sec. 7.1)

The universe U_f is fibrant.

Proof idea: Paths in the universe can be transformed into weak equivalences between their endpoints. Now given a partial path $T\colon (\Gamma,\varphi) \to U^{\mathbb{I}}$ and a total extension A of $T(1)$, by glueing we get a total type G which is a total extension of $T(0)$:

$$
\begin{tikzcd}
T(0) \arrow[rr, dashed, "\mathrm{glue}(\varphi,A,T,w)"] \arrow[dr, "w", "\sim"'] \arrow[ddr, bend right] & & G \arrow[dr, dashed, "\mathrm{unglue}(\varphi,A,T,w)", "\sim"'] \arrow[ddr, dashed] & \\
& T(1) \arrow[rr, hook, "\iota_{\varphi,A}"] \arrow[d] & & A \arrow[d] \\
& (\Gamma,\varphi) \arrow[rr, hook, "\iota_{\varphi}"'] & & \Gamma
\end{tikzcd}
$$

Concrete instances

Our (re)construction of the cubical set model has not required too much about the site $\mathbb{C}$ and the interval object $\mathbb{I}$. Nevertheless, the only known examples are the **algebraic theories** of **distributive lattices** and **de Morgan algebras**, resp.

Coquand has pointed out that in these particular cases the construction of the cubical set model can be performed in a fairly weak meta-theory, e.g. ETT with a sufficiently well-behaved universe.

This allows one to construct a whole bunch of new models replacing **Set** by models of sufficiently well-behaved models of ETT.

Modelling CTT in internal presheaves

Let $\mathcal{E}$ be a model of ETT with a universe $\mathcal{U}$ containing a natural numbers object (**nno**) and exact quotients of $\neg\neg$-closed equivalence relations.

Let $\mathcal{C}$ be the category internal to $\mathcal{E}$ which is the opposite of the category of finitely presented free de Morgan algebras and homomorphisms.

Then our interpretation of CTT carries over to the category of internal presheaves $\mathcal{E}^{\mathcal{C}^{\mathrm{op}}}$ since we have never made any substantial use of the subobject classifier in $\widehat{\mathbb{C}}$.

The universe U_f is impredicative in $\mathcal{E}^{\mathcal{C}^{\mathrm{op}}}$ whenever $\mathcal{U}$ is impredicative.

Realizability models for CTT

Thus, in particular, we can perform the construction of the cubical set model within $\mathcal{E} = \mathbf{Asm}(\mathcal{A})$ for any **pca** $\mathcal{A}$ instantiating $\mathcal{U}$ with the universe $\mathbf{Mod}(\mathcal{A})$.

Since $\mathbf{Mod}(\mathcal{A})$ is impredicative in $\mathcal{E}$ the ensuing universe $\mathcal{U}_f$ in $\mathcal{E}^{\mathcal{C}^{\mathrm{op}}}$ is impredicative as well (cf. also recent work by Awodey, Frey and Hofstra [Awo17, Fre17]).

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