Unfolding FOLDS

HoTT/UF Workshop

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Matthew Weaver and Dimitris Tsementzis

Princeton

Rutgers

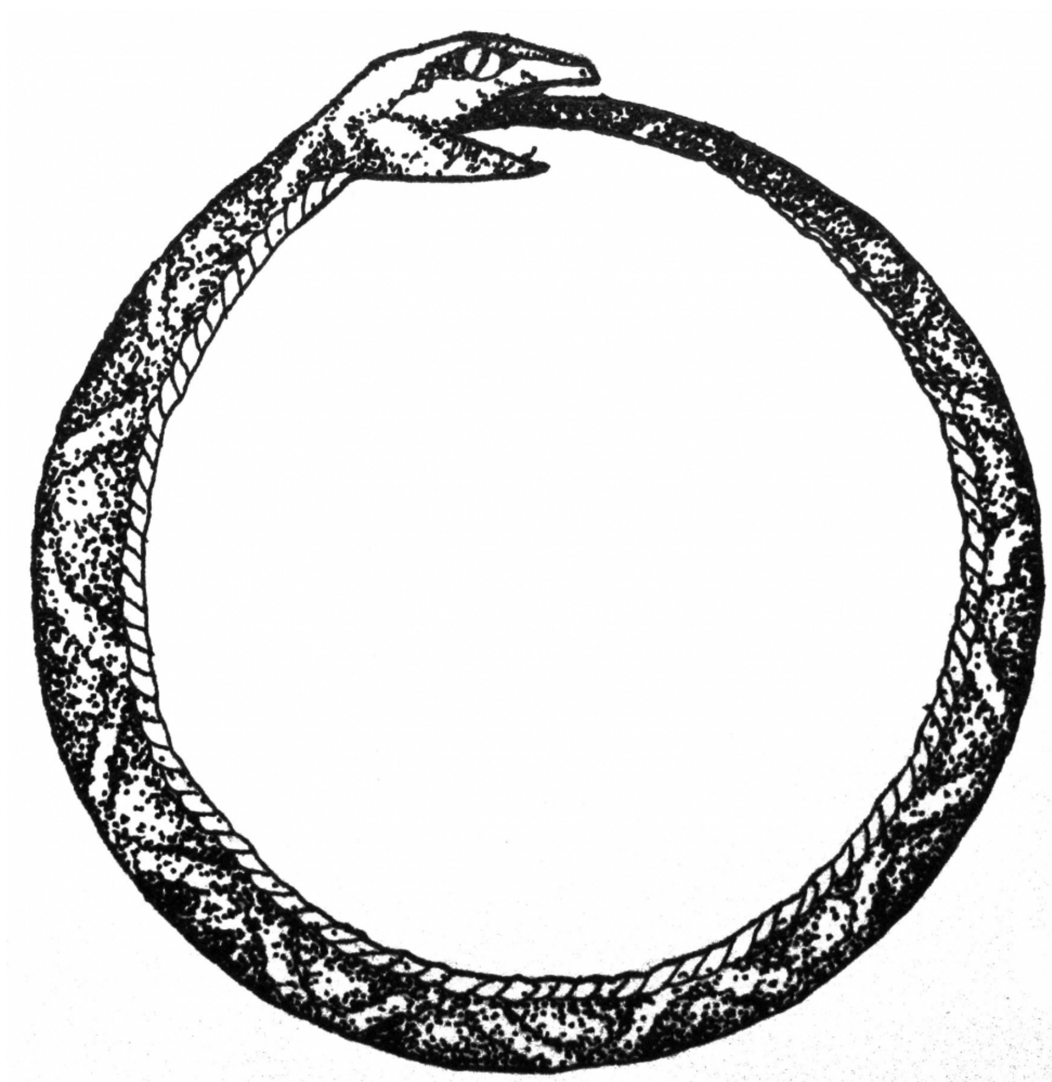
The Syntax of Syntax

- Type theory has a rich syntax…
- …which is why we love it!
- …and is also what makes everything difficult

The Syntax of Syntax

- We often encounter the situation where we can define a construct in the metatheory, but not internally
- **Challenge:** Let’s make type theory express its own metatheory
- **Bonus Challenge:** Let’s do so in a way that is well-typed and preserves logical consistency

Let's make type theory eat itself!



The Syntax of Syntax

- Meta-programming and reflection are already everywhere

Tactic languages in proof assistants:

```
Definition PreShv_to_slice_is_funct : is_functor PreShv_to_slice_data.
Proof.
  split; [intros X | intros X Y Z f g];
    apply eq_mor_slicecat;
    apply (nat_trans_eq has_homsets_HSET);
    unfold PreShv_to_slice_ob_nat , PreShv_to_slice_ob_funct_fun;
    intro c;
    apply funextsec; intro p;
    now rewrite tppr.
Defined.
```

The Syntax of Syntax

- Meta-programming and reflection are already everywhere

Generic programming over datatypes:

```
data Tm = Var TName
        | App Tm Tm
        | Lam (Bind (TName, Embed Tm) Tm)
        | Pi  (Bind (TName, Embed Tm) Tm)
        | Type
        | Kind
        deriving (Show, Generic, Typeable)
```

The Syntax of Syntax

- Meta-programming and reflection are already everywhere

Reflection of abstract syntax:

```
idNat : Nat -> Nat
idNat = %runElab (do intro `{{x}}
                     fill (Var `{{x}})
                     solve)
```

The Syntax of Syntax

- Meta-programming and reflection are already everywhere

Classical mathematics:

$$\frac{d}{dx}x^n = nx^{n-1}$$

The Syntax of Syntax

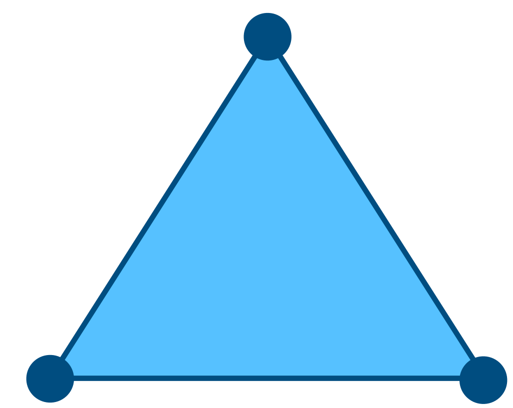
- In many cases, it is an untrusted extension of the theory that can break its good properties

The Syntax of Syntax

- We define a univalent type theory that can safely manipulate and interpret (some of) its own syntax
- Using this, we propose a novel approach to defining the type of semi-simplicial types
- We also describe a general framework to describe the semantics of reflection in type theory

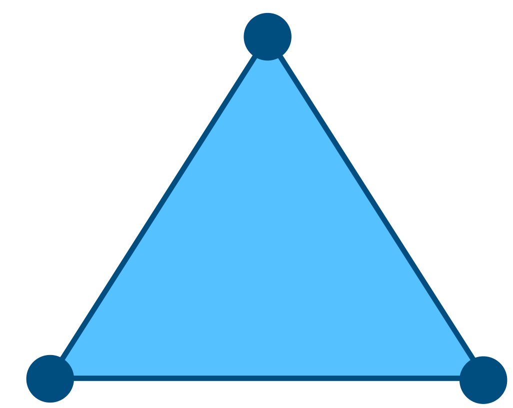
What are Semi-Simplicial Types?

- A “0-dimensional triangle” is a point
- A “1-dimensional triangle” is a line
- A “2-dimensional triangle” is a triangle
- A “3-dimensional triangle” is a pyramid/ tetrahedron made from 4 triangles, etc…



What are Semi-Simplicial Types?

- Consider a type of points T_0
- For any two terms (i.e. points) x and y in T_0, there is a type $T_1\ x\ y$ of lines between x and y
- For any three points x, y and z, and three lines $a : T_1\ x\ y$, $b : T_1\ y\ z$ and $c : T_1\ x\ z$, there is a type $T_2\ a\ b\ c$ of triangles outlined by a, b and c
- etc…

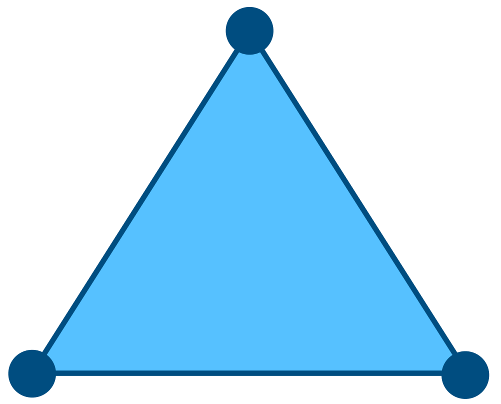


What are Semi-Simplicial Types?

$\Sigma\ T_0 : \mathrm{Type},$

$\Sigma\ T_1 : (\Pi\ x\ y : T_0, \mathrm{Type}),$

$\Sigma\ (T_2 : \Pi\ (x\ y\ z : T_0)\ (a : T_1\ x\ y)\ (b : T_1\ y\ z)\ (c : T_1\ x\ z), \mathrm{Type}),$



etc…

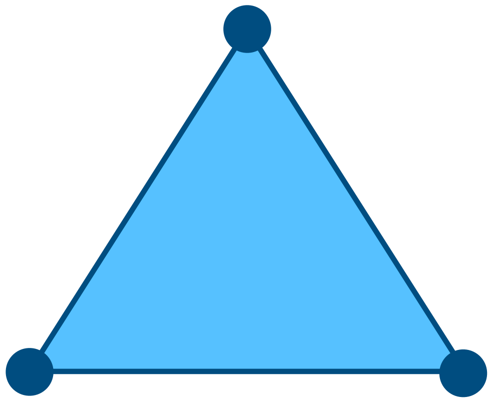
What are Semi-Simplicial Types?

- The type of *n*-truncated semi-simplicial types (sst_n) is given by $\Sigma T_0, \Sigma T_1, \ldots, T_n$
- It is a known result that type of semi-simplicial types is the homotopy limit of sst_n over $n : \mathbb{N}$ [ACS15]
- The homotopy limit is constructed with the following syntax where where π_n is the obvious projection from sst_{n+1} to sst_n:

$$\sum_{(x : \Pi_{(n:\mathbb{N})} \mathrm{sst}_n)} \prod_{(n:\mathbb{N})} \pi_n x_{n+1} = x_n$$

What are Semi-Simplicial Types?

- Defining the function sst : $\mathbb{N} \to$ Type picking out the n-truncated semi-simplicial type proves challenging:
  - All the dependencies in the types require proving equalities on terms of arbitrary types…
  - …which require proving equalities on proofs of equalities of terms of arbitrary types…
  - …and then proving equalities on proofs of equalities of proofs of equalities of terms of arbitrary types…
  - …etc…



What is FOLDS?

- First Order Logic with Dependent Sorts: FOL where sorts can be indexed by elements of other sorts (i.e. dependent types)

- A FOLDS Signature is a context of dependent sorts (equivalently a Finite Inverse Category)

- **Example:** Cat

  $O : \mathrm{Sort}$
  $A : O \times O \to \mathrm{Sort}$
  $I \ : \Pi\, x : O,\ A\ x\ x \to \mathrm{Sort}$

- **Note:** The type of *n*-truncated semi-simplicial types is such a signature with a sort for each dimension

Our Theory

- TT+I is a type theory that includes:
  - Π-types, Σ-types, id-types, $\mathbb{N}$, 1
  - A type Sig of FOLDS signatures
  - An interpretation function I : Sig → Type

Our Theory

- The type Sig of well-formed FOLDS signatures is built using the following:

  - Sig : Type is a list of well-formed contexts Ctx, each representing a sort by its dependencies

  - Ctx : Sig → Type is a list of sorts previously defined in the signature

- **Example:** representing Cat : Sig

  Cat ≔ O : ·, A : (c : O, d : O), I : (x : O, i : A x x)

Our Theory

- The type Sig of well-formed FOLDS signatures is built using the following:
    - Sig : Type is a list of well-formed contexts Ctx, each representing a sort by its dependencies
    - Ctx : Sig → Type is a list of sorts previously defined in the signature
    - …a couple other helper types
- Sig and Ctx are both h-sets (along with the other types)
- Complete definition is a quotient-inductive-inductive type in Agda à la type theory in type theory [AK16]

Our Theory

- The type Sig of well-formed FOLDS signatures is built using the following:
  - Sig : Type is a list of well-formed contexts Ctx, each representing a sort by its dependencies
  - Ctx : Sig → Type is a list of sorts previously defined in the signature
- The interpretation function I : Sig → Type is defined as

$$I(\Gamma_0, \Gamma_1, \ldots, \Gamma_n) \coloneqq \Sigma(T_0 : [\![\Gamma_0]\!] \to \mathrm{Type}), \Sigma(T_1 : [\![\Gamma_1]\!] \to \mathrm{Type}), \ldots, [\![\Gamma_n]\!] \to \mathrm{Type}$$

Our Theory

- The interpretation function $I : Sig \to Type$ is defined as

  $$I(\Gamma_0, \Gamma_1, \ldots, \Gamma_n) \coloneqq \Sigma(T_0 : [\![\Gamma_0]\!] \to Type), \Sigma(T_1 : [\![\Gamma_1]\!] \to Type), \ldots, [\![\Gamma_n]\!] \to Type$$

- **Example:** representing Cat in Sig

  $$Cat \coloneqq O : \cdot, A : (c : O, d : O), I : (x : O, i : A\ x\ x)$$

  $$I(Cat) \coloneqq \Sigma(O : Type), \Sigma(A : O \times O \to Type), (\Sigma(x : O), A(x, x)) \to Type$$

Defining Semi-Simplicial Types

1. Define sst' : $\mathbb{N}$ → Sig, picking out the n-truncated semi-simplicial type leveraging the strictness of Sig and Ctx
2. sst : $\mathbb{N}$ → Type ≔ I ∘ sst'
3. $$\sum_{(x : \Pi_{(n:\mathbb{N})} \mathrm{sst}_n)} \prod_{(n:\mathbb{N})} \pi_n x_{n+1} = x_n$$

How is this Reflection?

- Sig is a datatype representing the abstract syntax of the types corresponding to well-formed FOLDS signatures
- I is the interpretation function decoding terms of Sig into the types they represent
- **Note:** we only decode representations of terms, and never encode actual terms

So, what does this theory even mean?

Decoding the Universe(s)

- **(Informal) Definition:** A *universe (à la Tarski)* consists of a type U along with a decode function el : U → Type

- Our type Sig with interpretation function I is such a universe!

Decoding the Universe(s)

- **(incomplete) Definition:** Fix a category $\mathcal{C}$. A category with families (CwF) is a model of type theory with contexts given by $\mathcal{C}$ described by the following data:

- A presheaf $\mathrm{Ty} : \mathcal{C}^{op} \to \mathbf{Set}$, where $\mathrm{Ty}(\Gamma)$ is the set of all well-formed types in context Γ

- A presheaf $\mathrm{Tm} : \int\mathrm{Ty}^{op} \to \mathbf{Set}$, where $\mathrm{Tm}(\Gamma, A)$ is the set of all well-typed terms of type A in context Γ

    - $\int\mathrm{Ty}$ is the category of elements of Ty, consisting of pairs of contexts and well-formed types in that context

Decoding the Universe(s)

- **Definition:** Fix a CwF with presheaves $\mathrm{Ty} : \mathcal{C}^{op} \to \mathrm{Set}$, and $\mathrm{Tm} : \int \mathrm{Ty}^{op} \to \mathrm{Set}$. A *universe* is given by
    - A presheaf $\mathrm{U} : \mathcal{C}^{op} \to \mathrm{Set}$,
    - A decoding natural transformation $\mathrm{el} : \mathrm{U} \to \mathrm{Ty}$,
    - Types $\mathrm{U}_\Gamma \in \mathrm{Ty}(\Gamma)$ for every $\Gamma \in \mathcal{C}$ where $\mathrm{Tm}(\Gamma, \mathrm{U}_\Gamma) = \mathrm{U}(\Gamma)$ and the action of morphisms on the U_Γ is given by U
- Definition appears as 2-level CwF derived from a universe in Paolo's thesis [Cap17]

So, what does this theory even mean?

…we’ve also defined a 2-level type theory.

Decoding the Universe(s)

- While this notion of a universe can express adding a second universe with a strict equality on its codes of types, it doesn't provide a way to model strictness on (representations of) terms
- Captures Sig, but not the other types used to build Sig/ their strictness

From Universes to Reflection: (Idealized) Semantics

- Definition: A *type theory with reflection* is given by
  - a category with families $(\mathrm{Ty}, \mathrm{Tm})$ with universe $(\mathrm{U}, \mathrm{el})$
  - a presheaf $\mathrm{R} : \int \mathrm{U}^{\mathrm{op}} \to \mathrm{Set}$
  - a natural transformation $\mathrm{i} : \mathrm{el}[\mathrm{R}] \to \mathrm{Tm}$
    - Here $\mathrm{el}[-]$ denotes the functor $(\int \mathrm{U}^{\mathrm{op}} \to \mathrm{Set}) \to (\int \mathrm{Ty}^{\mathrm{op}} \to \mathrm{Set})$ induced by el
  - elements $\mathrm{A}_\Gamma \in \mathrm{Ty}(\Gamma)$ for every $\Gamma \in \mathcal{C}$ and $\mathrm{A} \in \mathrm{U}(\Gamma)$ such that $\mathrm{Tm}(\Gamma, \mathrm{A}_\Gamma) = \mathrm{R}(\Gamma, \mathrm{A})$

From Universes to Reflection: (Idealized) Semantics

- Haven't yet worked out if/how TT+I is a model of a type theory with reflection
- Part of what makes Sig powerful is it has an inductor. Not (yet) generalized in semantics I've proposed
  - The presence of an inductor is often assumed when one thinks of reflection of abstract syntax in general
- Can this be used to model more extensive (and safe!) reflection of abstract syntax in (univalent) type theory?

Connection to 2-Level Type Theory

- 2-Level Type Theory begins with MLTT+Axiom-K, and adds a second univalent universe that decodes into MLTT
    - MLTT+Axiom-K has two equality types: the strict one with axiom-K, and the one used to decode the equality of the univalent universe
- We begin with HoTT and add a second strict universe that decodes into HoTT
    - We have a single univalent equality type

Some Future Work

- Finish defining TT+I, implement it and define the type of semi-simplicial types
- Investigate simpler theories that can define the type of semi-simplicial types and only have one notion of equality
- Investigate whether definition of type theory with reflection makes any sense
    - If so, see what other interesting theories it models

Takeaways

- Make a (nonstandard) universe in which difficult problems are easy!
- Safe/consistent reflection in (univalent) type theory is both interesting and possible

References

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[AK16] Thorsten Altenkirch and Ambrus Kaposi, *Type theory in type theory using quotient inductive types*, ACM SIGPLAN Notices **51** (2016), no. 1, 18–29.

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