Lifting Problems in a Grothendieck Fibration

Andrew Swan

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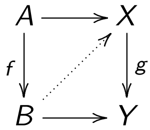

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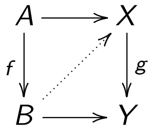

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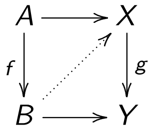

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We say f has the *left lifting property against* g and g has the *right lifting property against* f and write $f \pitchfork g$ if every lifting problem has a filler.
We say a class of maps $\mathcal{R}$ is *cofibrantly generated* if for some fixed set I, g belongs to $\mathcal{R}$ precisely when $f \pitchfork g$ for all $f \in I$.

For example, let $\mathbb{C} := \top$ the category of topological spaces, and let $f := e_0 \colon 1 \hookrightarrow [0,1]$ be an endpoint inclusion into the unit interval.

A map $g \colon X \to Y$ has the *path lifting property* if it has the right lifting property against e_0.

Examples include the map $\mathbb{R} \to \mathbb{S}^1$ given by $t \mapsto (\cos(t), \sin(t))$, and (trivially) the unique map $X \to 1$ for any topological space X.

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Another example of a cofibrantly generated class is Kan fibrations in simplicial sets, as used in the simplicial set model of type theory.

Grothendieck fibrations give an abstract formulation of the idea of "indexed families of objects." Before seeing the definition, we consider an example.

Definition

Let $\mathbb{C}$ be any category. We write $\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C})$ for the following category.

An object of $\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C})$ consists of any set I, together with a family of objects $(X_i)_{i \in I}$ with each X_i an object of $\mathbb{C}$.

A morphism from $(X_i)_{i \in I}$ to $(Y_j)_{j \in J}$ is a function $\sigma \colon I \to J$ together with a family of morphisms $(f_i)_{i \in I}$, with each f_i a morphism $X_i \to Y_{\sigma(i)}$ in $\mathbb{C}$.

Write $p\colon \mathbf{Fam}_{\mathrm{Set}}(\mathbb{C}) \to \mathrm{Set}$ for the forgetful functor sending $(X_i)_{i\in I}$ to I.

For a fixed I, we have a subcategory $\mathbf{Fam}_{\mathrm{Set}}(\mathbb{C})_I$ with objects in the preimage of I (families indexed by I) and morphisms in the preimage of the identity 1_I.
We call this the *fibre category over* I.

Note the maps f over 1_I in Set, are just families of maps $f_i : X_i \to Y_i$ for each $i \in I$. We refer to these as *vertical morphisms*.

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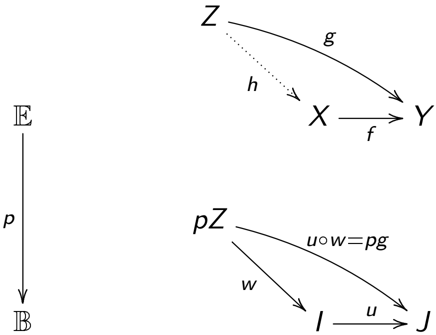
Given any function $\sigma\colon I \to J$ and a family $(Y_j)_{j\in J}$, we have a family over I defined by "reindexing": $\sigma^*(Y) := (Y_{\sigma(i)})_{i\in I}$.

In fact this lifts to a functor $\mathbf{Fam}_{\mathrm{Set}}(\mathbb{C})_J \to \mathbf{Fam}_{\mathrm{Set}}(\mathbb{C})_I$.

We also get a morphism $\bar{\sigma}(Y) : \sigma^*(Y) \to Y$ over σ, which is just the identity $1\colon Y_{\sigma(i)} \to Y_{\sigma(i)}$ for each $i \in I$.

Definition (Grothendieck)

Let $p\colon \mathbb{E} \to \mathbb{B}$ be a functor. A morphism $f : X \to Y$ in $\mathbb{E}$ is *cartesian over* $u : I \to J$ in $\mathbb{B}$ if $p(f) = u$ and for every $g : Z \to Y$ for which there exists w with $p(g) = u \circ w$, there is a unique $h : Z \to X$ in $\mathbb{E}$ such that $p(h) = w$ and $f \circ h = g$. In a diagram:



A morphism is cartesian over $\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C})$ iff it is isomorphic to some reindexing morphism $\bar{\sigma}(Y)$.

Definition (Grothendieck)

Let $p\colon \mathbb{E} \to \mathbb{B}$ be a functor. We say p is a (cloven) (Grothendieck) fibration if for every morphism $\sigma\colon I \to J$ in $\mathbb{B}$ and every $Y \in \mathbb{E}_J$ we have a choice of object $\sigma^*(Y)$ over I together with a cartesian map $\bar{\sigma}\colon \sigma^*(Y) \to Y$ over σ.

Example

$\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C}) \to \mathsf{Set}$ is a Grothendieck fibration with reindexing as given before.

Example

A map in the arrow category $\mathbb{C}^{\to}$ is cartesian over the codomain functor $\mathrm{cod}\colon \mathbb{C}^{\to} \to \mathbb{C}$ if and only if it is a pullback (as a square in $\mathbb{C}$).

(So this is a fibration if and only if $\mathbb{C}$ has all pullbacks.)

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Example

A *category indexed family* is a small category $\mathcal{A}$ together with a functor $X\colon \mathcal{A} \to \mathbb{C}$. A morphism from $X\colon \mathcal{A} \to \mathbb{C}$ to $Y\colon \mathcal{B} \to \mathbb{C}$ is a functor $\chi\colon \mathcal{A} \to \mathcal{B}$ together with a natural transformation $\alpha\colon X \Rightarrow Y \circ \chi$.

This forms a fibration $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C}) \to \mathsf{Cat}$ (with cartesian maps given by composition).

The fibre category over $\mathcal{A} \in \mathsf{Cat}$ is the functor category $[\mathcal{A}, \mathbb{C}]$.

Definition

Let $f\colon X \to Y$ be a vertical map over $I \in \mathbb{B}$ and $g\colon W \to Z$ a vertical map over $J \in \mathbb{B}$. A *family of lifting problems* of f against g consists of

1. An object K of $\mathbb{B}$.
2. Maps $\sigma\colon K \to I$ and $\tau\colon K \to J$ in $\mathbb{B}$.
3. A commutative square of vertical maps over K, of the following form:

$$\begin{array}{ccc} \sigma^*(X) & \longrightarrow & \tau^*(W) \\ {\scriptstyle \sigma^*(f)}\downarrow & & \downarrow{\scriptstyle \tau^*(g)} \\ \sigma^*(Y) & \longrightarrow & \tau^*(Z) \end{array}$$

A *solution*, or *family of fillers* of the lifting problems consists of a diagonal map $\sigma^*(Y) \to \tau^*(W)$ making two commutative triangles.

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$$\begin{array}{ccc} X_{\sigma(k)} & \longrightarrow & W_{\tau(k)} \\ {\scriptstyle f_{\sigma(k)}}\downarrow & & \downarrow{\scriptstyle g_{\tau(k)}} \\ Y_{\sigma(k)} & \longrightarrow & Z_{\tau(k)} \end{array}$$

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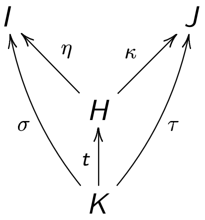
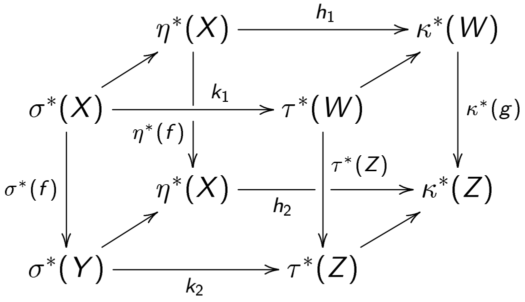
$$\begin{array}{ccc} X_{\sigma(k)} & \longrightarrow & W_{\tau(k)} \\ {\scriptstyle f_{\sigma(k)}} \downarrow & \overset{j_k}{\nearrow} & \downarrow {\scriptstyle g_{\tau(k)}} \\ Y_{\sigma(k)} & \longrightarrow & Z_{\tau(k)} \end{array}$$

A solution is a choice of diagonal filler j_k for each k.

Definition

Let $H, \eta, \kappa, h_1, h_2$ be a family of lifting problems of f against g. $H, \eta, \kappa, h_1, h_2$ is *universal* if for any family of lifting problems $K, \sigma, \tau, k_1, k_2$, there is a unique map $t \colon K \to H$ making the following diagrams commute (with the maps between the squares the unique ones over t given by cartesianess of $\bar{\eta}(f)$ and $\bar{\kappa}(g)$).





Theorem

Suppose the universal family of lifting problems from f to g exists. The following are equivalent.

1. *Every family of lifting problems of f against g has a solution.*
2. *The universal family of lifting problems has a filler.*
3. *There is a coherent choice of fillers for all families of lifting problems.*

Theorem

Let $p\colon \mathbb{E} \to \mathbb{B}$ be a fibration. Suppose further that $\mathbb{B}$ has all finite limits and p is locally small. Then all universal lifting problems exist and can be described explicitly in terms of Hom *objects over p.*

This applies to set indexed families and category indexed families on $\mathbb{C}$ precisely when $\mathbb{C}$ is locally small, and applies to codomain fibrations on $\mathbb{C}$ precisely when $\mathbb{C}$ is locally cartesian closed.

Note that any map $g \colon W \to Z$ in $\mathbb{C}$ can be viewed as a family of maps indexed by 1 in $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ and $\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C})$, or as a map in the slice category $\mathbb{C}/1$.

Definition

We say a class of morphisms $\mathscr{X}$ in $\mathbb{E}_1$ is *cofibrantly generated* if for some $I \in \mathbb{B}$ and some map $f \colon X \to Y$ in $\mathbb{E}_I$, $g \in \mathscr{X}$ if and only if the universal lifting problem of f against g has a solution.

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We say a category $\mathbb{D}$ with functor $\mathbb{D} \to \mathbb{E}_1^{\to}$ is *cofibrantly generated* if $\mathbb{D}$ is isomorphic over $\mathbb{E}_1^{\to}$ to the category consisting of pairs (g, j) where g is a map in $\mathbb{E}_1$ and j is a solution to the universal lifting problem against f.

Example (Quillen)

Applying this to set indexed families and assuming the axiom of choice recovers the classical notion of cofibrantly generated by Quillen.
This includes Kan fibrations in simplicial sets, Serre fibrations in topological spaces and many other examples.

Example (Garner)

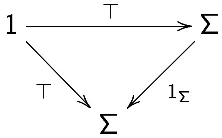
Applying this to category indexed families recovers Garner's notion of *algebraic lifting problem* and *algebraically cofibrantly generated*.

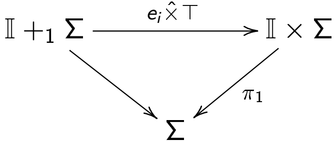
In turn this includes set indexed family cofibrantly generated as a special case as well as Kan fibrations and trivial fibrations in BCH cubical sets and CCHM cubical sets, and many other examples from homotopical algebra and elsewhere.

Example (Van den Berg and Frumin, Orton and Pitts)

Van den Berg and Frumin consider the following classes of maps. Let $\mathscr{D}$ be a class of maps closed under pullback (e.g. all monomorphisms). Let $\mathscr{X}$ be maps with the rlp against every element of $\mathscr{D}$. Let $e_0, e_1 : 1 \rightrightarrows \mathbb{I}$ be an interval object in $\mathbb{C}$ and let $\mathscr{Y}$ be the class of maps with rlp against $e_i \hat{\times} f$ for each $f \in \mathscr{D}$ and $i \in \{0, 1\}$.

Suppose there exists a universal map $\top : 1 \to \Sigma$ in $\mathscr{D}$ (i.e. every map in $\mathscr{D}$ is a pullback of $\top$). Then $\mathscr{X}$ and $\mathscr{Y}$ are both cofibrantly generated w.r.t. the codomain fibration by the following families of maps over Σ:





By writing out the universal lifting problems explicitly we see this is the same as a class of maps studied by Orton and Pitts.

Theorem

Let p be a bifibration (i.e. $p^{\mathrm{op}} \colon \mathbb{E}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is also a fibration) and suppose it is locally small. Fix a vertical map f in $\mathbb{E}$. Then we can factorise every vertical map $g \colon W \to Z$ as $W \xrightarrow{L_1 g} \cdot \xrightarrow{R_1 g} Z$ such that:

1. *The factorisation is functorial over $\mathbb{B}$.*
2. *R_1 induces a pointed endofunctor and L_1 induces a comonad on vertical maps in $\mathbb{E}$.*
3. *R_1-Algebra structures on g correspond precisely to solutions of the universal lifting problem of g against f.*

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- In most cases $R_1 g$ itself is not necessarily a right map.
- Applying this to category indexed families recovers step 1 of Garner's small object argument.
- It is constructed from Hom objects and opcartesian maps, which are uniquely determined up to isomorphism for a given fibration.

Theorem (Garner)

Let $\mathbb{C}$ be a cocomplete category and suppose that for every $X \in \mathbb{C}$ the functor $\mathbb{C}(X, -)$ preserves α-filtered colimits for some regular cardinal α.
Then, for each vertical map f we get a functorial factorisation (L, R) over category indexed families as before, such that:

1. *R induces a monad and L induces a comonad on vertical maps in $\mathbb{E}$.*
2. *R-Algebra structures on g correspond precisely to solutions of the universal lifting problem of g against f.*

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- Since R is a monad, Rg is a right map for any g.
- This is a slight extension of Garner's result (which only constructs the factorisation for maps in $\mathbb{C}$).
- Gitik has proved it is consistent with **ZF** that ω is the only regular cardinal.

Theorem (S)

Let $\mathbb{C}$ be a topos with natural number object satisfying WISC. Consider the codomain fibration on $\mathbb{C}$. Then, for each vertical map f the algebraic weak factorisation system cofibrantly generated by f exists.

Remark

WISC (weakly initial set of covers) is a choice axiom, but a very weak one.
Van den Berg and Moerdijk have proved that WISC is stable under a large number of constructions and in particular holds in all Grothendieck toposes, all realizability toposes and internal presheaves in realizability toposes (assuming it holds in the background theory).

Garner's argument is a transfinite iteration of step 1 along a regular ordinal α.

One way to phrase the construction is "freely adding a filler for every lifting problem." This involves adding elements for the image of the filler, and then quotienting to make the upper triangle commute and ensure that every filler is only added once. Adding new elements and quotienting both add new lifting problems.

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- Ordinals are difficult to use internally in a topos, so instead we use W-types which have well understood categorical semantics.
- Alternating between adding new elements and quotienting is difficult to do with W-types, so instead we add all elements in one go, and then quotient once at the end.
- By using WISC we can ensure that we are now finished and no new elements have to be added after quotienting.

The general framework allows us to compare the new internal constructions with preëxisting work.

Observation

In Garner's notion of cofibrantly generated, the generating cofibrations are a functor $\mathcal{A} \to \mathbb{C}^{\to}$ and the solutions to the lifting problem need to be natural with respect to morphisms in $\mathcal{A}$ (this is referred to as *uniformity*). Nothing like this appears in the work of Van den Berg and Frumin and Pitts and Orton.

Explanation

The base of the codomain fibration is just objects of $\mathbb{C}$ with no addition structure. This makes it closer to $\mathbf{Fam}_{\mathsf{Set}}(\mathbb{C})$ than $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ in some ways.

We could also consider internal versions of $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ where the base consists of internal categories in $\mathbb{C}$, but this is unnecessary - many models of type theory including the CCHM model can be understood using cod directly.

Observation

Over $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ we have to choose the generating cofibrations carefully (making essential use of uniformity) to satisfy Gambino and Sattler's requirement that cofibrations are algebraically stable under pullback. This is not required by Van den Berg and Frumin or Pitts and Orton.

Explanation

Pullback stability is inherent in the codomain fibration, and stability of cofibrations is closely linked to coherence. One can show under certain conditions that if the generating family of cofibrations is a map into the terminal object of its slice category, then cofibrations are stable under pullback.

Observation

Garner's definitions (and older notions due to Quillen) refer to a notion of set and category which are external to $\mathbb{C}$. In particular, Garner's small object argument makes essential use of the existence of all small (set sized) colimits in $\mathbb{C}$. A non trivial colimit appears already in step one of the small object argument.

Explanation

The base of the category indexed family fibration consists of all small categories, most of which have nothing to do with $\mathbb{C}$, whereas the base of the codomain fibration is just $\mathbb{C}$ itself.

The opcartesian maps in $\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ are given by left Kan extension, whereas the opcartesian maps in cod are simply given by composition.

Observation

Garner's results work with many categories that are not necessarily cartesian closed whereas dependent products play an essential role in the codomain fibration.

Explanation

There is a general notion of Hom object in fibrations, such that Hom objects are unique if they exist. If all Hom objects exist we say the fibration is *locally small*.

$\mathbf{Fam}_{\mathsf{Cat}}(\mathbb{C})$ is locally small if and only if $\mathbb{C}$ is locally small, with Hom objects defined as comma categories.

cod is locally small if and only if $\mathbb{C}$ is locally cartesian closed with Hom objects defined as local exponentials.

Observation

Results by Sattler and Gambino refer to a monoidal product on $\mathbb{C}$ that can be, but doesn't need to be cartesian product. Results by Van den Berg and Frumin and Pitts and Orton only apply to cartesian product.

Explanation

Any monoidal product on $\mathbb{C}$ lifts to a fibred monoidal product over category indexed families, which is simply defined pointwise.

Monoidal products that are fibred over the codomain fibration are more difficult to construct, with the exception of cartesian product which extends to a fibred product simply given by pullback.

Summary:

- A general notion of lifting problem in Grothendieck fibrations can be applied to category indexed families or codomain fibrations to give short definitions of many interesting classes of maps.
- Step 1 of the small object argument applies in any locally small bifibration and is uniquely determined up to isomorphism.
- Cofibrantly generated algebraic weak factorisation systems exist over the codomain fibration of many interesting categories.

Thank you for your attention!