A coinductive approach to ∞-equivalence relations

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Overview & Goals

- To formulate in homotopy type theory precise criteria of what counts as ‘the structure of an equivalence relation’.
- To give an example of such a structure

  $$\mathsf{isEqRel} : \prod_{(A:\mathcal{U})} \mathsf{rRel}_A \to \mathcal{U}$$

  that meets the criteria of the first goal.
- To specify in homotopy type theory the structure of being a loop space.

- A reflexive relation on a type A consists of

$$R : A \to A \to \mathcal{U}$$

  and a proof of reflexivity

$$\rho : \prod_{(a:A)} R(a, a).$$

- The type of all (small) reflexive relations on A is written

$$\mathrm{rRel}_A.$$

- Given a map $f : A \to X$, we define the prekernel $k(f)$ of f to be the reflexive relation

$$a, b \mapsto f(a) = f(b)$$

  on A.

- A map $f : A \to X$ is called surjective if

$$\prod_{(x:X)} \|\mathsf{fib}_f(x)\|.$$

- The type of all surjective maps out of A is

$$(A \downarrow_s \mathcal{U}) :\equiv \sum_{(X:\mathcal{U})} \sum_{(f:A\to X)} \mathsf{isSurj}(f)$$

- If we restrict the prekernel operation to the surjective maps, we get

$$(A \downarrow_s \mathcal{U}) \xrightarrow{k} \mathsf{rRel}_A$$

Problem

To define a structure of homotopy coherent equivalence relations

$$\mathsf{isEqRel} : \prod_{(A:\mathcal{U})} \mathsf{rRel}_A \to \mathcal{U}$$

such that for each $A : \mathcal{U}$

- *the prekernel operation* k *has a lift*

$$\begin{array}{ccc} & & \sum_{(\mathcal{R}:\mathsf{rRel}_A)} \mathsf{isEqRel}_A(\mathcal{R}) \\ & \overset{\mathcal{K}}{\nearrow} & \downarrow \\ (A \downarrow_s \mathcal{U}) & \xrightarrow[k]{} & \mathsf{rRel}_A. \end{array}$$

  Call $\mathcal{K}(f)$ *the kernel of* f.

- *The map* $\mathcal{K}$ *is an equivalence.*

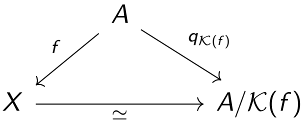
- The lift $\mathcal{K}$ ensures that every prekernel comes equipped with the structure of an equivalence relation.
- The condition that $\mathcal{K}$ is an equivalence gives:
  - an inverse operation

    $$\mathcal{Q} : \left(\sum_{(\mathcal{R}:\mathsf{rRel}_A)} \mathsf{isEqRel}_A(\mathcal{R})\right) \to (A \downarrow_s \mathcal{U}).$$

    In other words, it assigns to every equivalence relation $\mathcal{R}$

    a quotient type $A/\mathcal{R}$ and a quotient map $q_{\mathcal{R}} : A \to A/\mathcal{R}$.

  - By the homotopy $\mathcal{Q} \circ \mathcal{K} \sim \mathsf{id}$ it follows that for every surjective map $f : A \to X$ we have



- - By the homotopy $\mathcal{K} \circ \mathcal{Q} \sim \mathrm{id}$ it follows that for every equivalence relation $\mathcal{R} \equiv (R, \rho, e)$ we have

$$k(q_{\mathcal{R}}) = (R, \rho).$$

  Furthermore, the proof that $k(q_{\mathcal{R}})$ is an equivalence relation is equal to e.

  In other words, the homotopy $\mathcal{K} \circ \mathcal{Q} \sim \mathrm{id}$ is precisely the effectivity of the quotienting operation $\mathcal{Q}$.

$$\mathsf{isEqRel}_A :\equiv \mathsf{fib}_k$$

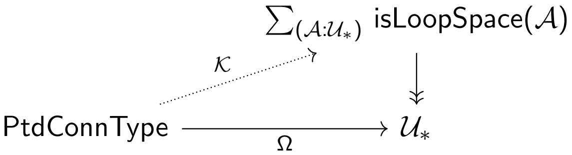
Problem

To define a homotopy coherent loop space structure

$$\mathsf{isLoopSpace} : \mathcal{U}_* \to \mathcal{U}$$

such that

- *the loop space operation Ω has a lift*

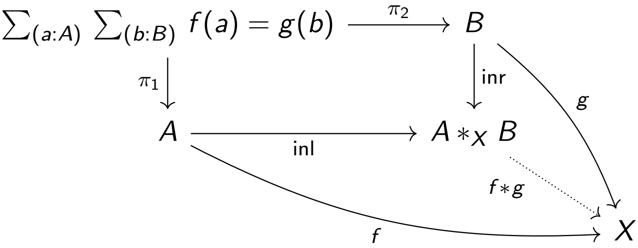


  so that every loop space comes equipped with the structure of being a loop space.

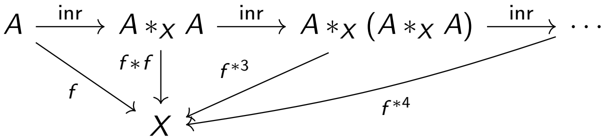
- *The map $\mathcal{K}$ is an equivalence.*

Definition

Let $f : A \to X$ and $g : B \to X$ be maps with a common codomain. The join $f * g$ of maps is defined by first pulling back, and then pushing out the pullback span:



- In the join construction we take the join-powers f^{*n} of maps. This gives rise to a sequence



  that converges to the image inclusion of f.
- The image of f is a model for the ∞-quotient $A/k(f)$.

Theorem

Consider a map $f : A \to X$ and the join $f * f : A *_X A \to X$ of f with itself. Then the commuting square

$$\begin{array}{ccc}
\sum_{(a,b:A)} f(a) = f(b) & \dashrightarrow & \sum_{(x,y:A *_X A)} (f * f)(x) = (f * f)(y) \\
\downarrow & & \downarrow \\
A \times A & \xrightarrow[\mathsf{inr} \times \mathsf{inr}]{} & (A *_X A) \times (A *_X A)
\end{array}$$

is a pullback square.

In this sense, the pre-kernel of $f * f$ extends the pre-kernel of f.

Every pre-kernel extends to a new pre-kernel.

Definition

Consider a type A, with a reflexive relation (R, ρ) over it. A reflexive graph quotient of the reflexive graph (A, R, ρ) consists of a type X that comes equipped with

$$\alpha_0 : A \to X$$
$$\alpha_1 : \prod_{(a,b:A)} R(a,b) \to (\eta_0(a) = \eta_0(b))$$
$$\alpha_r : \prod_{(a:A)} \eta_1(\rho(a)) = \mathsf{refl}_{\eta_0(a)}$$

and satisfies the corresponding induction principle.

We assume every reflexive graph (A, R, ρ) has a reflexive graph quotient

$$\mathsf{rcoeq}(A, R, \rho).$$

Theorem

Let Γ be a reflexive graph, and let X be a type, and let

$$\begin{aligned}
\alpha_0 &: A \to X \\
\alpha_1 &: \prod_{(a,b:A)} R(a,b) \to (\eta_0(a) = \eta_0(b)) \\
\alpha_r &: \prod_{(a:A)} \eta_1(\rho(a)) = \mathsf{refl}_{\eta_0(a)}.
\end{aligned}$$

Then the following are equivalent:

1. *X satisfies the induction principle of the reflexive graph quotient.*
2. *The square*

$$\begin{array}{ccc}
\sum_{(x,y:\Gamma_0)} \Gamma_1(x,y) & \xrightarrow{\pi_2} & \Gamma_0 \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \alpha_0} \\
\Gamma_0 & \xrightarrow[\alpha_0]{} & X
\end{array}$$

   is a pushout square.

Every pre-kernel extends to a pre-kernel on its own reflexive graph quotient.

Lemma

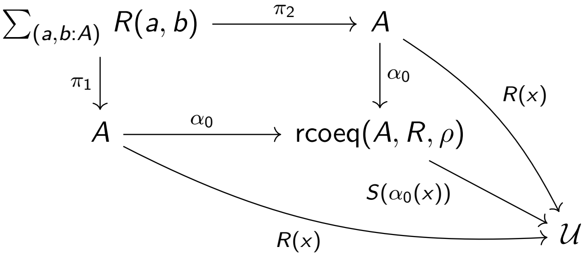
Let (R,ρ) *be a reflexive relation on* A*, and suppose* (S,σ) *is a reflexive relation on* $\mathsf{rcoeq}(A,R,\rho)$ *such that the canonical family of maps*

$$\prod_{(a,b:A)} R(a,b) \to S(\alpha_0(a),\alpha_0(b))$$

is a family of equivalences (i.e. S *extends* R*). Then we have*

$$\prod_{(x,a,b:A)} R(a,b) \to (R(x,a) \simeq R(x,b))$$

Proof.


$\sum_{(a,b:A)} R(a,b)$
π_2
A
π_1
α_0
$R(x)$
A
α_0
$\mathrm{rcoeq}(A, R, \rho)$
$S(\alpha_0(x))$
$R(x)$
$\mathcal{U}$

□

Definition

Let $\mathcal{R} \equiv (R, \rho)$ be a reflexive relation on a type A. We say that $\mathcal{R}$ has the structure of a hereditarily reflexive relation if there is a term of the indexed coinductive type $\mathsf{isHRR}(\mathcal{R})$ consisting of

1. reflexive relations $\mathcal{S}$ on $\mathsf{rcoeq}(A, R, \rho)$ that extend $\mathcal{R}$,
2. such that $\mathcal{S}$ again has the structure of a hereditarily reflexive relation.

Given a hereditarily reflexive relation $\mathcal{R}$ on A, we obtain a sequence

$$(A_0, R_0, \rho_0) \longrightarrow (A_1, R_1, \rho_1) \longrightarrow (A_2, R_2, \rho_2) \longrightarrow \cdots$$

of reflexive graphs and reflexive graph morphisms, where $(A_0, R_0, \rho_0) \equiv (A, R, \rho)$, and

$$A_{n+1} :\equiv \mathrm{rcoeq}(A_n, R_n, \rho_n),$$

and (R_{n+1}, ρ_{n+1}) is obtained from the fact that (R_n, ρ_n) is a hereditarily reflexive relation.

Let $\mathcal{R}$ be a hereditarily reflexive relation on A. Then we define $A/\mathcal{R}$ to be the sequential colimit of

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

with $q_{\mathcal{R}}$ defined to be the map $A_0 \to A/\mathcal{R}$ of the cocone structure of $A/\mathcal{R}$.

Lemma

The map $q_{\mathcal{R}}$ is surjective

This defines the quotienting operation $\mathcal{Q}$.

Lemma

Let $f : A \to X$ be a map. Then the pre-kernel $k(f)$ can be given the structure of a hereditarily reflexive relation, defining the operation $\mathcal{K}$.

Theorem

Let $f : A \to X$ be a map. Then we have a commuting triangle

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{q_{\mathcal{K}(f)}} \\ \mathrm{im}(f) & \xrightarrow[\simeq]{\qquad\qquad} & A/\mathcal{K}(f) \end{array}$$

in which the bottom map is an equivalence. Hence we have $\mathcal{Q} \circ \mathcal{K} \sim \mathrm{id}$.

Lemma

Let $\mathcal{R}$ be a hereditarily reflexive relation on A. Using the fact that each R_{n+1} extends R_n, we can define

$$R_\infty : A_\infty \to A_\infty \to \mathcal{U}.$$

such that $R(i_n(a), i_n(b)) \simeq R_n(a, b)$ for each $a, b : A_n$.

Theorem

For each $a : A$, the total space

$$\sum\nolimits_{(x:A_\infty)} R_\infty(i_0(a), x)$$

is contractible. Hence (R, ρ) is the pre-kernel of $q_{\mathcal{R}}$, and thus the triangle

$$\begin{array}{ccc} & & \sum_{(\mathcal{R}:\mathsf{rRel}_A)} \mathsf{isEqRel}_A(\mathcal{R}) \\ & \swarrow^{\mathcal{Q}} & \downarrow \\ (A \downarrow_s \mathcal{U}) & \xrightarrow[k]{} & \mathsf{rRel}_A. \end{array}$$

commutes.

Conclusion

- It remains to lift the homotopy of the previous theorem to a homotopy $\mathcal{K} \circ \mathcal{Q} \sim \mathrm{id}$.
- We hope to do this using a slightly different definition of hereditarily reflexive relations, that extends relations only on one argument.
- Our theorems are in the process of being formalized in the Coq proof assistant.