Homotopy Type-Theoretic Interpretations of CZF

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Outline

1. Type-theoretic interpretations of CZF
2. New interpretations $[\![\cdot]\!]_{\infty,2}$ and $[\![\cdot]\!]_{1,2}$
3. Proof-theoretic characterisation of the formulas valid in $[\![\cdot]\!]_{\infty,2}$

Part I:
Type-theoretic interpretations of CZF

Constructive Set Theory (CZF)

The underlying logic is intuitionistic rather than classical.

Axioms:

- Extensionality
- Pairing
- Union
- Emptyset
- Infinity
- $\in$-Induction
- Bounded Separation

  $\forall a \, \exists x \, \forall y (y \in x \leftrightarrow y \in a \wedge \phi(y))$ for ϕ bounded

- Strong Collection

  $\forall x \in a \, \exists y \phi(x, y) \rightarrow \exists b (\forall x \in a)(\exists y \in b)\phi(x, y) \wedge (\forall y \in b)(\exists x \in a)\phi(x, y)$

- Subset Collection

Aczel's Interpretation

1. Interpretation of **sets**: $\mathcal{V} := (W\, A : \mathcal{U})A$

   The introduction rule is:

   $$\frac{A : \mathcal{U} \qquad f : A \rightarrow \mathcal{V}}{sup(A, f) : \mathcal{V}}$$

2. Interpretation of **equality**.
   Given $A, B : \mathcal{U}$ and $f : A \rightarrow \mathcal{V}$ and $g : B \rightarrow \mathcal{V}$ we define
   $[\![sup(A, f) \doteq sup(B, g)]\!]$ as:

   $$(\Pi x : A)(\Sigma y : B)[\![f(x) \doteq g(y)]\!] \times (\Pi y : B)(\Sigma x : A)[\![f(x) \doteq g(y)]\!]$$

Aczel's Interpretation

3. The interpretation of the other **formulas** follows the propositions-as-types correspondence:

$$[\![\phi \wedge \psi]\!] := [\![\phi]\!] \times [\![\psi]\!]$$
$$[\![\phi \to \psi]\!] := [\![\phi]\!] \to [\![\psi]\!]$$
$$[\![\phi \vee \psi]\!] := [\![\phi]\!] + [\![\psi]\!]$$

$$[\![(\forall x \in sup(A, f))\phi(x)]\!] := (\Pi x : A)[\![\phi(f(x))]\!]$$
$$[\![(\exists x \in sup(A, f))\phi(x)]\!] := (\Sigma x : A)[\![\phi(f(x))]\!]$$

$$[\![\forall x\ \phi(x)]\!] := (\Pi \alpha : \mathcal{V})[\![\phi(\alpha)]\!]$$
$$[\![\exists x\ \phi(x)]\!] := (\Sigma \alpha : \mathcal{V})[\![\phi(\alpha)]\!]$$

$$[\![\alpha \in \beta]\!] := [\![(\exists y \in \beta)(y \doteq \alpha)]\!]$$

Other Interpretations

HoTT book interpretation, 2013.

- the type $\mathcal{V}$ is a higher inductive type, $Id_{\mathcal{V}}$ interprets equality and is defined as a bisimulation relation
- $\mathcal{V}$ is a hset, however $\mathcal{V}$ itself is constructed with arbitrary small types
- formulas are interpreted as hpropositions

Remark:
Univalence is invoked in the proof of a lemma that gives a description the terms of $\mathcal{V}$.
An inspection of the proofs shows that univalence is not needed overall.

Other Interpretations

Gylterud's PhD thesis, 2016.

- interpretation of a *multiset* theory in type theory, sets are special multisets
- multisets are interpreted as small types
- formulas are interpreted as hprops
- the identity type interprets equality

Part II:
New interpretations

New Interpretation $[\![\cdot]\!]_{\infty,2}$: sets-as-hsets

1. Idea: **sets** are interpreted as hsets. Define $hSet_{\mathcal{U}} := (\Sigma A : \mathcal{U}) ishSet(A)$. Then:

$$\mathcal{V}_2 := (W\, x : hSet_{\mathcal{U}})\pi_1(x)$$

   We work in the type theory that uses just the rules for $\mathcal{V}_2$, namely: $ML_1^i \mathcal{V}_2 + FE$.

2. Interpretation of **equality**. As in Aczel's interpretation define $[\![sup(A, f) \doteq sup(B, g)]\!]_{\infty,2}$ as:

$$(\Pi x : A)(\Sigma y : B)[\![f(x) \doteq g(y)]\!]_{\infty,2} \times (\Pi y : B)(\Sigma x : A)[\![f(x) \doteq g(y)]\!]_{\infty,2}$$

New Interpretation $[\![\cdot]\!]_{\infty,2}$: sets-as-hsets

3. Interpretation of the other **formulas**, as in Aczel's interpretation:

$$[\![\phi \wedge \psi]\!]_{\infty,2} := [\![\phi]\!]_{\infty,2} \times [\![\psi]\!]_{\infty,2}$$
$$[\![\phi \to \psi]\!]_{\infty,2} := [\![\phi]\!]_{\infty,2} \to [\![\psi]\!]_{\infty,2}$$
$$[\![\phi \vee \psi]\!]_{\infty,2} := [\![\phi]\!]_{\infty,2} + [\![\psi]\!]_{\infty,2}$$

$$[\![(\forall x \in sup(A,f))\phi(x)]\!]_{\infty,2} := (\Pi x : A)[\![\phi(f(x))]\!]_{\infty,2}$$
$$[\![(\exists x \in sup(A,f))\phi(x)]\!]_{\infty,2} := (\Sigma x : A)[\![\phi(f(x))]\!]_{\infty,2}$$

$$[\![\forall x\ \phi(x)]\!]_{\infty,2} := (\Pi \alpha : \mathcal{V}_2)[\![\phi(\alpha)]\!]_{\infty,2}$$
$$[\![\exists x\ \phi(x)]\!]_{\infty,2} := (\Sigma \alpha : \mathcal{V}_2)[\![\phi(\alpha)]\!]_{\infty,2}$$

$$[\![\alpha \in \beta]\!]_{\infty,2} := [\![(\exists y \in \beta)(y \doteq \alpha)]\!]_{\infty,2}$$

Theorem

All axioms of CZF are valid in the interpretation $[\![\cdot]\!]_{\infty,2}$.

New Interpretation $[\![\cdot]\!]_{1,2}$: sets-as-hsets, formulas-as-hprops

1. Interpretation of **sets** as hsets: $\mathcal{V}_2 := (W\,x : hSet_{\mathcal{U}})\pi_1(x)$
2. Interpretation of **equality**, we define $[\![sup(A, f) \doteq sup(B, g)]\!]_{1,2}$ as:

$$(\Pi x : A)\ \|(\Sigma y : B)[\![f(x) \doteq g(y)]\!]_{1,2}\| \times (\Pi y : B)\ \|(\Sigma x : A)[\![f(x) \doteq g(y)]\!]_{1,2}\|$$

3. Interpretation of the other **formulas**:

$$[\![\phi \vee \psi]\!]_{1,2} := \|[\![\phi]\!]_{1,2} + [\![\psi]\!]_{1,2}\|$$
$$[\![(\exists x \in sup(A, f))\phi(x)]\!]_{1,2} := \|(\Sigma x : A)[\![\phi(f(x))]\!]_{1,2}\|$$
$$[\![\exists x\ \phi(x)]\!]_{1,2} := \|(\Sigma \alpha : \mathcal{V}_2)[\![\phi(\alpha)]\!]_{1,2}\|$$

Theorem

The following axioms are valid in the interpretation $[\![\cdot]\!]_{1,2}$: Extensionality, Pairing, Union, Emptyset, Infinity, $\in$-Induction, Bounded Separation.

Part III:
Proof-theoretic characterisation of $[\![\cdot]\!]_{\infty,k}$

Proof-theoretic Aspects of $[\![\cdot]\!]_{\infty,k}$

Definition
A *CC formula* is one in which no unbounded quantifier occurs in the antecedent of an implication.

CC formulas include all formulas used in standard practice.

Definition

- A set is a *base* iff it satisfies the axiom of choice.
- A set is *ΠΣ-generated* iff it is generated from finite sets and ω using the operations Π the set of dependent functions and Σ the disjoint union.

ΠΣ-**AC**: every ΠΣ-generated set is a base.

Theorem (Rathjen & Tupailo)

Let ϕ be a CC sentence.
Then $CZF + \Pi\Sigma\text{-}AC \vdash \phi$ if and only if $ML_1^e\mathcal{V} \vdash t : [\![\phi]\!]$ for some term t.

Proof-theoretic Aspects of $[\![\cdot]\!]_{\infty,k}$

Lemma
$\Pi\Sigma$-AC is valid in $[\![\cdot]\!]_{\infty,k}$.

Theorem
Given a CC sentence ϕ, if $ML_1^i\mathcal{V}_k + FE \vdash s : [\![\phi]\!]_{\infty,k}$ for some term s, then $CZF + \Pi\Sigma\text{-}AC \vdash \phi$.

Proof.
It follows from the existence of an interpretation of $ML_1^i\mathcal{V}_k + FE$ into $ML_1^e\mathcal{V}$ such that $[\![\cdot]\!]_{\infty,k}$ is mapped to $[\![\cdot]\!]$. □

Proof-theoretic Aspects of $[\![\cdot]\!]_{\infty,k}$

Corollary

Given a CC sentence ϕ of CZF and two numbers $h, k \in \mathbb{N} \cup \{+\infty\}$ with $2 \leq h, k$. We have:

$$\begin{array}{ccc} ML_1^i\mathcal{V}_k + FE \vdash t : [\![\phi]\!]_{\infty,k} & \Leftrightarrow & ML_1^i\mathcal{V}_h + FE \vdash s : [\![\phi]\!]_{\infty,h} \\ \textit{for some term } t & & \textit{for some term } s \end{array}$$

Future Work

We have different interpretations of CZF in type theory. They interpret sets, formulas and equality in different ways.

However, they share a core structure.

Idea
Provide an analysis of these type-theoretic interpretations using a logic enriched type theory.

References

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