WEAK EQUIVALENCES BETWEEN CATEGORIES OF MODELS OF TYPE THEORY

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It is conjectured that intensional type theory is the internal language of higher categories, in a manner similar to how extensional type theory is the internal language of 1-categories. Kapulkin and Lumsdaine [KL16] have outlined a specific program by which to precisely state and prove such a conjecture. They exhibit a functor Cl_∞ from the category $\mathsf{CxlCat}_{\Sigma,\mathsf{Id}}$ of contextual categories (with sum and identity types) into the category Lex_∞ of finitely complete quasicategories, which they conjecture to be a Dwyer-Kan (DK-) equivalence that further lifts to a DK-equivalence between $\mathsf{CxlCat}_{\Sigma,\mathsf{Id},\Pi}$ (that is, $\mathsf{CxlCat}_{\Sigma,\mathsf{Id}}$ with product types) and the category LCCC_∞ of locally cartesian closed quasicategories.

There are different ways to model type theory in categories, each of which is more syntactic or more homotopical in flavor to varying degrees. On the more syntactic side, [KL16] establishes an adjuction between $\mathsf{CxlCat}_{\Sigma,\mathsf{Id},\Pi}$ and the category $\mathsf{CwA}_{\Sigma,\mathsf{Id},\Pi}$ of categories with attributes, as well as a notion of weak equivalence for each of these categories. Categories with attributes are essentially split comprehension categories, and [LW15] exhibits an adjunction between the category $\mathsf{SplCompCat}_{\Sigma,\mathsf{Id},\Pi}$ of split comprehension categories and the category $\mathsf{ClvCompCat}_{\Sigma,\mathsf{Id},\Pi}$ of cloven comprehension categories. On the more homotopical side, [Shu15] introduces the notion of type-theoretic fibration category (the category of which we denote $\mathsf{TTFibCat}$), which in particular are simultaneously comprehension categories and fibration categories.

All the categories involved have homotopy-theoretic structure, which can be presented by giving the category the structure of a category with weak equivalences. We establish notions of weak equivalence for (the categories of) comprehension categories and type-theoretic fibration categories, and show that the comparison functors relating the various categories of models of type theory mentioned above are all homotopy equivalences, building on previous work towards the conjecture. We thus establish a precise, homotopy-theoretic sense in which the different ways to model type theory are all equivalent. In particular we have the following:

**Theorem.** *There is a Dwyer-Kan equivalence between* $\mathsf{CxlCat}_{\Sigma,Id,\Pi}$ *and* $\mathsf{TTFibCat}$.

The results of [Szu14] imply that Lex_∞ is DK-equivalent to the category FibCat of fibration categories, which should lift to a suitable extension of FibCat with extra structure. Establishing a DK-equivalence between $\mathsf{CxlCat}_{\Sigma,\mathrm{Id},\Pi}$ and this extension FibCat^* would imply the conjecture as stated in [KL16]. Our result therefore reduces the conjectured equivalence between $\mathsf{CxlCat}_{\Sigma,\mathrm{Id},\Pi}$ and LCCC_∞ to showing an equivalence between $\mathsf{TTFibCat}$ and FibCat^*, and lifting Szumiło's result.

The included diagram presents the structure of our proof, and situates our result in the context of the conjectured equivalence. The diagram presents, on the left, the adjunctions between the categories involved; and on the right, the various notions of weak equivalence within these categories, which we list below:

- *logical equivalence* (LE) represents a notion of bi-interpretability;
- *weak logical equivalence* (wLE), a non-strict version of logical equivalence;

- *weak contextual equivalence* (wCE), an *a priori* stronger notion of bi-interpretability;

- *homotopical equivalence* (HE), the natural notion of equivalence between fibration categories.

The horizontal equalities in the diagram indicate that the notions of weak equivalence within the category are the same, while the vertical arrows indicate a homotopy equivalence (and hence a DK-equivalence) between the categories, equipped with their respective weak equivalences.
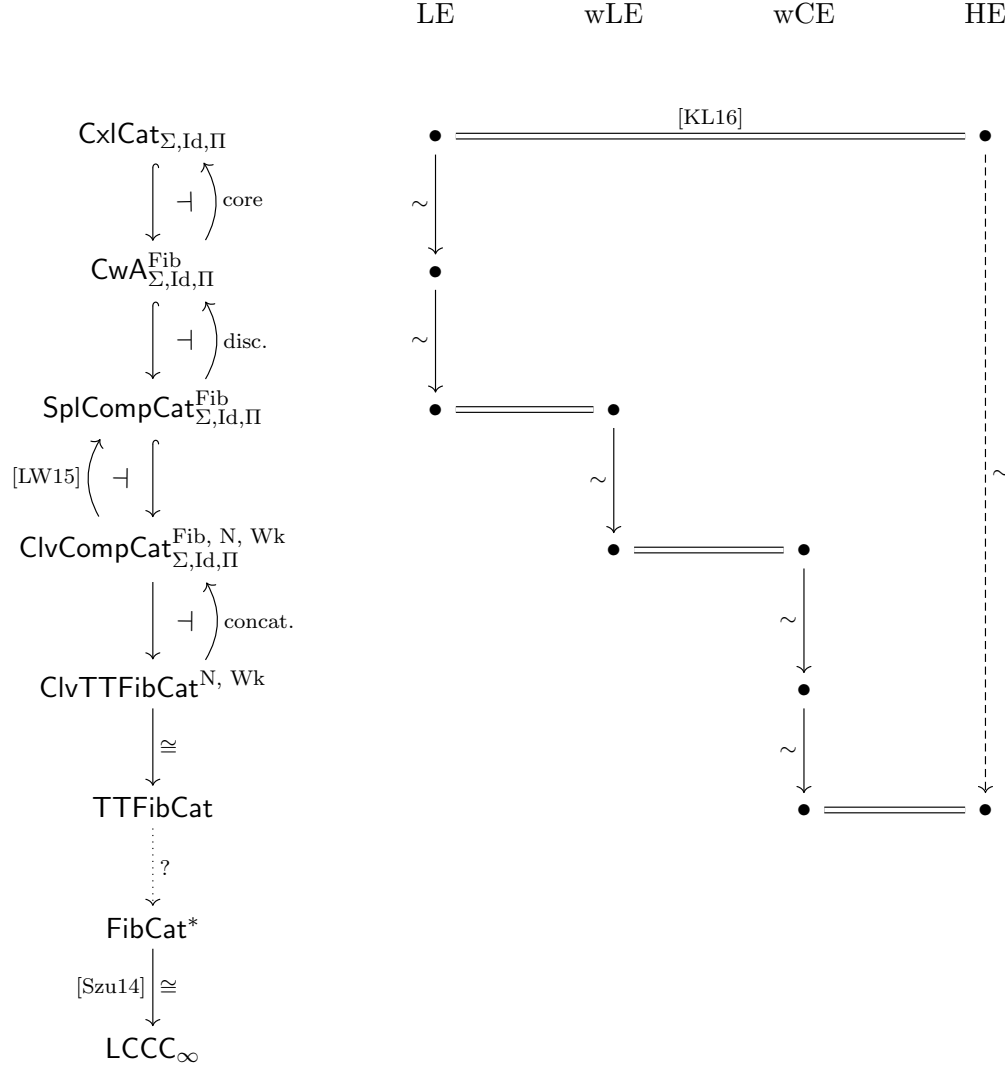
References

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