Interpreting Type Theory in Appropriate Presheaf Toposes

Thomas Streicher, Jonathan Weinberger
TU Darmstadt, Germany

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In [CCHM16] Coquand et al. have presented a model for a variant of Martin-Löf type theory called *Cubical Type Theory* (CTT) which allows one to prove Voevodsky's *Univalence Axiom* and establish the existence of various *Higher Inductive Types*. The model they describe lives within a presheaf topos of so-called cubical sets.

The aim of this talk is to reformulate this model in more categorical terms using the internal language of the ambient topos as much as possible. In this respect our approach is inspired by [OP16] where they work within an anonymous topos. We restrict to presheaf toposes since they allow for an explicit *construction* of a universe as opposed to *loc.cit.* where a universe is just postulated. This allows us to perform the so-called glueing construction as in [CCHM16] without postulating an ad hoc condition on the universe as in [OP16].

Possibly the approach of [GS17, Sat17] is equivalent to the one of [OP16] and ours after unfolding definitions in the internal language á la Kripke-Joyal.

In [CCHM16] the authors have given an interpretation of CTT in a presheaf topos $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ where $\mathbb{C}$ is the opposite of the category of free finitely generated de Morgan algebras. Though we will consider this particular site as a running example, we want to keep a certain level of generality in our exposition with the intention of identifying a minimal set of assumptions on $\mathbb{C}$ sufficient for obtaining a model of CTT. First of all $\mathbb{C}$ should have finite products. Moreover, there should be a distinguished object $\mathbb{I}$ with two distinct global elements 0 and 1 satisfying some axioms as in [OP16]. Also, there should be a subobject Cof of Ω which is a *dominance* in the sense of Rosolini [Ros86], i.e. the monos whose characteristic maps factor through Cof are closed under composition and contain all isomorphisms.

We will formulate our models of CTT as particular "categories with attributes" (Cartmell) *aka* "type categories" (Pitts) whose underlying category is $\widehat{\mathbb{C}}$. For this purpose we have to come up with a discrete fibration (presheaf) $\mathcal{T}$ over $\widehat{\mathbb{C}}$ together with a cartesian ("comprehension") functor $\mathfrak{C}$ from $\mathcal{T}$ to the fundamental (*aka* "codomain") fibration $P_{\widehat{\mathbb{C}}}$ of $\widehat{\mathbb{C}}$.

In [CCHM16] they have described their model externally without making use of the internal language of the topos $\widehat{\mathbb{C}}$ as opposed to [OP16]. We prefer this latter version because it allows one to formulate things more concisely. But for this purpose we need a universe U in $\widehat{\mathbb{C}}$ which can be constructed à la Hofmann and Streicher [Str05]. As is well known, $\widehat{\mathbb{C}}$ gives rise to a model of *extensional* type theory with a proof irrelevant impredicative universe as given by the subobject classifier Ω where equivalent propositions are already equal.

In this framework we define what is a *fibration structure* on a family of types A indexed over Γ following [OP16]. We will formulate this in the language of extensional type theory interpreted in $\widehat{\mathbb{C}}$ making use of the universe U and the impredicative universe Ω which should be interpreted in the model based on $\widehat{\mathbb{C}}$ as described above. Somewhat informally a fibration structure on A over Γ for every path $\gamma : \mathbb{I} \to \Gamma$ and every partial path p in $\Pi_{\mathbb{I}}(A \circ \gamma)$ with extent in Cof sends a totalization of p at $i \in \{0, 1\}$ to a totalization of p at $1 - i$. Thus, having a fibration structure on a family of types is not a local property, i.e. it cannot be formulated as a requirement on types which all items of the family validate. But this is in accordance with Kan fibrations which cannot be simply understood as a family of Kan objects. Despite this non-locality we can define a universe U_f whose generalized elements at stage Γ correspond to families of small types over Γ endowed with a fibration structure.

We also adapt the notion of a *glueing construction*, as considered in [CCHM16, OP16], to our generalized sites. As in [CCHM16] this glueing construction is crucial for showing that U_f validates Voevodsky's univalence axiom.

Finally, though not checked in every detail, we think that Gambino and Sattler's approach [GS17, Sat17] based on algebraic model structures may be considered as an externalization via Kripke-Joyal semantics of the respective internal definitions employed in [OP16] and our work.

References

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