MODEL STRUCTURE ON THE UNIVERSE IN A TWO LEVEL TYPE THEORY

SIMON BOULIER, NICOLAS TABAREAU

ABSTRACT. Last year we presented how to formalize a model structure on the universe of fibrant types in Homotopy Type System, and one on the universe of all types in a hypothetical system with a finer notion of fibrancy [8]. We come back this year with a concrete system allowing it!

1. A TYPE SYSTEM WITH TWO EQUALITIES AND A FIBRANT REPLACEMENT

Homotopy Type System (HTS) is a system introduced by Voevodsky [10] (and spelled out in [2]) which enjoys two notions of equality. The first one $\equiv$ is a strict equality, it enjoys functional extensionality and uniqueness of identity proofs (UIP). It reflects the mathematical equality of the simplicial or cubical model. The second one $=$ reflects the path equality in the model. It enjoys univalence (and thus functional extensionality). As univalence and UIP are contradictory [9], HTS requires a mechanism to prevent the strict equality and the univalent equality from collapsing. This is achieved by introducing the notion of *fibrant types* (the terminology comes from their interpretations in homotopical models). Thus, there is a new judgment $\Gamma \vdash A\ \ \mathbf{Fib}$ which expresses that a type is *fibrant*. All usual types are fibrant, except strict equality types. Then, the elimination of the univalent equality is restricted to fibrant types so that :

$$x \equiv y \to x = y \qquad\qquad \text{but} \qquad\qquad x = y \not\to x \equiv y$$

Given, such a judgment, it is natural to wonder if a fibrant replacement is admissible. A fibrant replacement is a modality $\overline{A}$ turning any type into a fibrant type with appropriated introduction and elimination rules.

$$\frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma\,;\, . \vdash \overline{A}\ \ \mathbf{Fib}}$$

Unfortunately, such fibrant replacement have been noticed to be inconsistent [1, 4] (this rely on the existence of a map $x = y \to \overline{x \equiv y}$). We thus propose a refinement of the notion of fibrancy of HTS to avoid this inconsistency.

We introduce $\mathrm{MLTT}_2^{\mathcal{F}}$, a system similar to HTS where the fibrancy judgment is replaced by $\Gamma\,;\,\Delta \vdash A\ \ \mathbf{Fib}$ (Δ is another context). We thus distinguish two levels of context. When this judgment is derivable, we say that, in the context Γ, the type family $\Delta \vdash A$ is *regularly fibrant*. In the case where only $\Gamma, \Delta\,;\, . \vdash A\ \ \mathbf{Fib}$ is derivable, we say that $\Delta \vdash A$ is *degenerately fibrant*—which is a weaker condition. Indeed, regular fibrancy implies degenerate fibrancy but the converse does not hold.

Fibrancy rules are similar to HTS ones. For instance, the rule for dependent product is:

$$\frac{\Gamma\,;\,\Delta \vdash A\ \ \mathbf{Fib} \qquad \Gamma\,;\,\Delta, x : A \vdash B\ \ \mathbf{Fib}}{\Gamma\,;\,\Delta \vdash \Pi\, x : A.\, B\ \ \mathbf{Fib}}$$

As in HTS, the only non fibrant types are strict equality types, and the fibrancy commutes with all other type constructors. The universes of types and fibrant types remain fibrant. The elimination rule for identity types is restricted to **regularly** fibrant predicates:

$$\frac{\Gamma \vdash A\ \mathbf{Fib} \qquad \Gamma \vdash t, t' : A \qquad \Gamma \vdash p : t =_A t' \qquad \Gamma ; y : A, q : t =_A y \vdash P\ \mathbf{Fib} \qquad \Gamma \vdash u : P\,\{y := t, q := \mathbf{refl}_t\}}{\Gamma \vdash J_{=}(A, y.q.P, t, t', p, u) : P\,\{y := t', q := p\}}$$

We then introduce a fibrant replacement in $\mathrm{MLTT}_2^{\mathcal{F}}$. The fibrant replacement is an operator that turns any type A into a **degenerately** fibrant type $\overline{A}$. Asking only for a degenerately fibrant replacement is the key to avoid inconsistency.

$$\frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma \vdash \overline{A} : \mathcal{U}_i} \qquad \frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma ; . \vdash \overline{A}\ \mathbf{Fib}} \qquad \frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma \vdash \eta_A : A \to \overline{A}}$$

$$\frac{\Gamma ; z : \overline{A} \vdash P(z)\ \mathbf{Fib} \qquad \Gamma \vdash t : \Pi\, x : A.\ P(\eta_A\, x)}{\Gamma \vdash \mathbf{repl\_ind}_P\ t : \Pi\, z : \overline{A}.\ P(z)} \qquad \mathbf{repl\_ind}_P\ t\,(\eta_A\, x) \simeq_{\beta\eta} t\, x$$

We implemented $\mathrm{MLTT}_2^{\mathcal{F}}$ in Coq using private inductive types and type classes.

2. Interpretation of $\mathrm{MLTT}_2^{\mathcal{F}}$ in the cubical model

We give an interpretation of $\mathrm{MLTT}_2^{\mathcal{F}}$ in the Bezem-Huber-Coquand cubical model without connections [3, 6]. In this model, a fibrant type of $\mathrm{MLTT}_2^{\mathcal{F}}$ is interpreted by a cubical set family equipped with a uniform degenerate Kan structure:

Definition 1. *Given a family* $\Gamma, \Delta \vdash A$ *, a* uniform degenerate Kan structure *over A relative to Γ is given by:*

- *for all $I \in \square$, S shape on I of direction x, $\rho \in \Gamma(I)$* ***degenerate along*** *x, $\delta \in \Delta(\rho)$ and $\vec{u}$ open-box of shape S in $A(\rho, \delta)$, a* filler $[A(\rho, \delta)]_S \vec{u} \in A(\rho, \delta)$
- *such that for all $(y, b) \in \langle S \rangle$, $([A\rho]_S \vec{u})(y = b) = u_{yb}$*
- *and such that for each $f : I \to I'$ with $J, x \subseteq \mathsf{def}(f)$ (ρf is thus degenerate along $f(x)$), $([A(\rho, \delta)]_S \vec{u}) f = [A(\rho f, \delta f)]_{Sf}(\vec{u} f)$*

The only difference with a bare Kan structure is that the quantification on the elements in the first part of the context is restricted to degenerate elements.

3. Application: model structure on the universes in $\mathrm{MLTT}_2^{\mathcal{F}}$

The fibrant replacement allow to define a model structure on the universes $\mathcal{U}_i$ of all types, and not only on the universes of fibrant types as in [5, 7]. The two weak factorization systems are given by:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\sim", "\lambda x.\,(f\,x, \eta\,x, \mathbf{refl})"'] & & B \\ & \Sigma_{y:B}\ \mathbf{fib}_{\overline{f}}\,(\eta\, y) \arrow[ur, "\pi_1"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda x.\,(f\,x, \mathbf{top}\,(\eta\, x))"'] & & B \\ & \Sigma_{y:B}\ \mathbf{Cyl}_{\overline{f}}\,(\eta\, y) \arrow[ur, "\sim", "\pi_1"'] & \end{tikzcd}$$

where $\mathbf{fib}_f\ y$ is $\Sigma\, x : A.\ f\, x = y$ the homotopy fiber in y, and $\mathbf{Cyl}_f\ y$ its mapping cylinder (see [7]).
We formalized this model structure in our implementation of $\mathrm{MLTT}_2^{\mathcal{F}}$ in Coq. The code is available at: `https://github.com/CoqHott/model-structures-Coq`.

REFERENCES

[1] The nLab wiki. `https://ncatlab.org/homotopytypetheory/show/Homotopy+Type+System`, 2016.

[2] Thorsten Altenkirch, Paolo Capriotti, and Nicolai Kraus. Extending homotopy type theory with strict equality. *Computer Science Logic*, 2016, Marseille, France.

[3] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. *Post-proceedings of TYPES 2013*, LIPIcs Volume 26, 2014.

[4] Paolo Capriotti. Models of type theory with strict equality. *PhD thesis*, 2017.

[5] Nicola Gambino and Richard Garner. The identity type weak factorisation system. *Theoretical Computer Science*, Volume 409, 2008.

[6] Simon Huber. Cubical intepretations of type theory. *PhD thesis*, 2016.

[7] Peter L.Lumsdaine. *Model Structures from Higher Inductive Types*. Unpublished notes, `http://peterlefanulumsdaine.com/research/Lumsdaine-Model-strux-from-HITs.pdf`, 2011.

[8] Nicolas Tabareau Simon Boulier. *Model structures on types in type theory*. talk at HoTT/UF 2016, Porto, `http://hott-uf.gforge.inria.fr/nicolas.pdf`, June 2016.

[9] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[10] Vladimir Voevodsky. *A simple type system with two identity types*. Unpublished notes, `http://uf-ias-2012.wikispaces.com/file/view/HTS.pdf`, 2013.