Modalities in homotopy type theory

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In traditional modal logic, a *modality* is a unary operation on propositions. The classical examples are $\Box$ ("it is necessary that") and $\Diamond$ ("it is possible that"). In type theory and particularly dependent type theory, such as homotopy type theory, where propositions are regarded as certain types, it is natural to extend the notion of modality to a unary operation on *types*. For emphasis we may call this a "typal modality", or a "higher modality" since it acts on the "higher types" available in homotopy type theory (not just "sets" but types containing higher homotopy).

We take a first step towards the study of higher modalities in *homotopy* type theory [3], restricting our attention to *idempotent, monadic* ones. These are especially convenient they can be described using the universal property of a reflector into a subuniverse. Another reason is that in good situations, an idempotent monad can be extended to all slice categories consistently, and thereby represented "fully internally" in type theory as an operation $\bigcirc : \mathsf{U} \to \mathsf{U}$ on a type universe. Our theory of modalities can be (and has been) formalized in the Coq proof assistant.

We give four equivalent characterizations of modalites: (1) *higher modalities*, (2) *uniquely eliminating modalities*, (3) Σ*-closed reflective subuniverses*, and (4) *stable orthogonal factorization systems*. For the purpose of this abstract, we give just the definition of a Σ-closed reflective subuniverse: A *reflective subuniverse* consists of a subuniverse $\mathsf{isModal} : \mathsf{U} \to \mathsf{Prop}$ and an operation $\bigcirc : \mathsf{U} \to \mathsf{U}$, and a modal unit $\eta : \prod_{(X:\mathsf{U})} X \to \bigcirc X$, subject to the following conditions:

(i) for any $X : \mathsf{U}$ one has $\mathsf{isModal}(\bigcirc X)$.

(ii) for any $X, Y : \mathsf{U}$ such that $\mathsf{isModal}(Y)$, the precomposition map

$$\lambda g.\, g \circ \eta_X : (\bigcirc X \to Y) \to (X \to Y)$$

is an equivalence.

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A reflective subuniverse is said to be Σ-*closed* if for any $X : \mathsf{U}$ such that $\mathsf{isModal}(X)$, and any $Y : X \to \mathsf{U}$ such that $\prod_{(x:X)} \mathsf{isModal} Y(x)$, one has

$$\mathsf{isModal}\Big(\textstyle\sum_{(x:X)} Y(x)\Big).$$

Examples of modalities include the *n-truncations*, for any proposition Q the *open modality* $X \mapsto X^Q$ and the *closed modality* $X \mapsto Q * X$ (where the *join* $Q * X$ is defined as the pushout of the span $Q \leftarrow Q \times X \to X$), and the *double negation modality* $X \mapsto \neg\neg X$.

An important class of reflective subuniverses is that of the *accessible reflective subuniverses*, which is presented as the subuniverse of types X that are F-local for a family of maps $F : \prod_{(a:A)} B(a) \to C(a)$, in the sense that for each $a : A$, the precomposition map

$$\lambda g.\, g \circ F(a) : (C(a) \to X) \to (B(a) \to X)$$

is an equivalence. The operations of *F-localization* is defined as a higher inductive type, which makes the subuniverse of F-local types into a reflective subuniverse. In the special case where each $C(a)$ is contractible, we call the localization operation *B-nullification*, and we show that B-nullification is always a higher modality by showing that the subuniverse of B-null types is Σ-closed.

Specializing further, we show that nullification at a family of *propositions* always gives a *lex modality*, which is a modality that preserves pullbacks. We give several equivalent characterizations of lex modalites. Among those is the condition that the subuniverse of modal types is again a modal type. Since general reflective subunverses are already closed under Σ, Π, and Id, this shows that lex modalities fully model type theory, including the universe. This is not the case for ordinary modalities (for instance, the type of all propositions is not a proposition).

Assuming that Prop is a small type, we obtain Lawvere-Thierney topologies as class of examples. In particular, the *double negation sheaf modality* is defined by nullifying at the propositions P for which $\neg\neg P$ holds.

We end the paper by proving a general "fracture and gluing" theorem for a pair of modalities, which has as a special case the "Artin gluing" of a complementary closed and open subtopos. Let $\lozenge$ be a lex modality, and let $\bigcirc$ be a modality such that the $\lozenge$-modal types coincide with the $\bigcirc$-connected types. Then we show that the trivial modality is the join of $\lozenge$ and $\bigcirc$ in the poset of reflective subuniverses, and that every *canonical fracture square*

$$\begin{array}{ccc} A & \xrightarrow{\eta^{\bigcirc}_A} & \bigcirc A \\ {\scriptstyle \eta^{\lozenge}_A}\big\downarrow & & \big\downarrow{\scriptstyle \eta^{\lozenge}_{\bigcirc A}} \\ \lozenge A & \xrightarrow[\lozenge \eta^{\bigcirc}_A]{} & \lozenge\bigcirc A \end{array}$$

associated to A is a pullback square. Moreover, we obtain an induced equivalence

$$\mathsf{U} \simeq \sum_{(B:\mathsf{U}_\bigcirc)} \sum_{(C:\mathsf{U}_\lozenge)} (C \to \lozenge B)$$

We call this a "fracture theorem" because it appears formally analogous to the fracture theorems for localization and completion at primes in classical homotopy theory [2], or more generally for localization at complementary generalized homology theories [1].

References

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