Lifting Problems in a Grothendieck Fibration

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The notion of *lifting problem* is a central concept in homotopical algebra, as well as in the semantics of homotopy type theory.

Given two maps $m\colon U \to V$ and $f\colon X \to Y$ in a category $\mathbb{C}$, we say that m has the *left lifting property* with respect to f and f has the *right lifting property* with respect to m if for every commutative square, as in the solid lines below (which we refer to as a *lifting problem*), there is a *diagonal filler*, which is the dotted line below, making two commutative triangles.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "m"'] & X \arrow[d, "f"] \\ V \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

In the presence of the axiom of choice, this is equivalent to having a choice of diagonal filler for every lifting problem of m against f.

A well known example is where we take $\mathbb{C}$ to be the category **Top** of topological spaces and m to be an endpoint inclusion into the unit interval $e_0\colon 1 \hookrightarrow [0,1]$. When a map f has the right lifting property against e_0 we say it has the *path lifting property*.

A *weak factorisation system* (wfs) is two classes of maps $\mathcal{L}$ and $\mathcal{R}$ which are closed under retracts and such that every element of $\mathcal{L}$ has the left lifting property against every element of $\mathcal{R}$.

Often it's useful to talk about not just individual maps, but to have some notion of indexed family of maps. The most basic example is where we have a set I and a family of maps $(m_i)_{i \in I}$. Then we say that f has the right lifting property against $(m_i)_{i\in I}$ if we have for every $i \in I$ and every lifting problem of m_i against f a choice of diagonal filler. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* if $\mathcal{R}$ is precisely the class of maps with the right lifting property against some fixed set indexed family of maps. A classic example from homotopical algebra is where we take $\mathbb{C}$ to be the category of simplicial sets and take $(m_i)_{i\in I}$ to be the set of horn inclusions $(\Lambda^n_k \hookrightarrow \Delta^n)_{n\in\mathbb{N}, 0 \leq k \leq n}$. Then if f has the right lifting property against the family of horn inclusions, we say that it is a *Kan fibration*.

A more sophisticated notion due to Garner is where we replace the indexing set, I with a small category $\mathcal{C}$. In this case, a family of maps is a functor $M\colon \mathcal{C} \to \mathbb{C}^{\to}$. In particular, this includes not just a choice of map $M(c)$ for each object c of $\mathcal{C}$, but also a choice of commutative square for each morphism in $\mathcal{C}$. Garner showed that in categories satisfying certain conditions, every such family cofibrantly generates not just a wfs, but the more structured notion of *algebraic weak factorisation system* (awfs). These ideas have seen much use in homotopical algebra, but are also important in the semantics for homotopy type theory. In the cubical set models of type theory due to Coquand et al, the dependent types are implemented using a variant of Kan fibration, where one has not just any choice of diagonal filler, but a choice which satisfies a "uniformity condition." This uniformity condition can then be formulated using Garner's notion of lifting problem.

When developing realizability semantics for homotopy type theory one is naturally lead to consider yet another notion of lifting problem. In certain categories, such as *presheaf assemblies* (presheaves constructed internally in a category of assemblies) it makes sense to say that one not only has a choice of diagonal filler, but that the filler can be found uniformly computably from $i \in I$ and lifting problem of m_i against f.

Finally, many people, including but not limited to Stekelenburg, Coquand, Pitts and Orton, Van den Berg and Frumin, have studied aspects of what amounts to a notion of lifting problem internal to a topos. This can be used to give an alternative understanding of the definition of Kan fibration in the Cohen-Coquand-Huber-Mörtberg model of cubical type theory (in addition to Garner's notion of algebraic lifting).

I will give a general notion of lifting problem which makes sense in any Grothendieck fibration. All of the examples above can then be recovered by applying the general definition to suitable examples of Grothendieck fibrations. If, in addition the fibration has the structure of a cloven locally small bifibration with finitely complete base, then we can carry out step 1 of the small object argument. This takes as input a "family of generating left maps" and gives us a left hand side of an awfs (lawfs) and thereby a pointed endofunctor on $\mathbb{C}^{\to}$ whose algebras consist of maps together with a choice of diagonal filler for the family of lifting problems against the generating left maps. A key idea is the use of local smallness to produce "universal lifting problems" between any two families of maps. As a consequence we obtain a precise definition of internal lifting problem in a topos and can see clearly how it relates to more standard notions. We can then in general state what it means for an algebraically free awfs to exist on the lawfs and show that it is equivalent to the existence of a choice of initial objects for a certain collection of categories. I will give a new version of the small object argument that constructs algebraically free awfs's in toposes with natural number object that satisfy a weak version of the axiom of choice, WISC (weakly initial set of covers). In contrast to many existing versions of the small object argument, cocompletness is not required. Consequentially the result applies to some interesting examples based on realizability where the underlying category is not cocomplete.