Homotopy Type-Theoretic Interpretations of CZF

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Abstract

In this talk I will present two variants of Aczel's type-theoretic interpretation of CZF expanding on what has been done on set theory in the HoTT book. The first one interprets sets as hsets, whereas the second one interprets sets as hsets and formulas as hpropositions. I will give a proof-theoretic characterisation of the formulas of CZF validated in the sets-as-hsets interpretation showing that CZF, in a sense, cannot distinguish between the sets-as-types and the sets-as-hsets interpretations.

Background

Constructive Set Theory (CZF) is a set theory introduced by Aczel in [Acz78] based on intuitionistic logic, which does not include the Power Set axiom. It is a predicative theory in the sense that it does not permit impredicative definitions, i.e. ones where a set is defined in terms of a set to which it belongs. For example the definition of closure in a topological space: "given a subset of a space $S \subseteq X$, the closure $\bar{S}$ is the smallest closed set containing S", the closure $\bar{S}$ is indeed a closed set containing S. Instead in CZF when an expression like "the smallest set such that" appears it is a shorthand for an authentically constructive procedure that generates the set from ones that are already constructed. In this respect CZF is similar to Martin-Löf type theory where no powertype operation is available and types are built from below from previously constructed types.

The relationship between CZF and Martin-Löf type theory is well studied ([Acz78], [RT06], [GR94]). In particular [Acz78] constructs an interpretation of CZF in Martin-Löf type theory which is a map that for any formula ϕ assigns a type $[\![\phi]\!]$ using the proposition-as-types correspondence. For every axiom of CZF one can construct a term inhabiting its interpretation. The key idea is that sets can be interpreted as trees where nodes represent sets and edges the membership relation so that the root represent the given set. This intuition is captured by interpreting the universe of sets as the W-type $V := (W x : U) x$. In this interpretation, since sets are interpreted as trees, set-theoretic equality is interpreted using a bisimulation relation on trees.

Motivation

The aim of the talk is to explore variants of Aczel's interpretation. For this recall that type-theoretic interpretations of set theory have three main degrees of freedom corresponding to the interpretations of: sets (as types, hsets), formulas (as types, hpropositions) and equality (as bisimulation or Id-type).

In the HoTT book (see [UFP13, section 10.5]) there is an interpretation of set theory in type theory which interprets sets as types, formulas as hpropositions, and the type V interpreting the universe of sets is a higher inductive type so that it is a hset and its identity type interprets set-theoretic equality.

In [Gy16a] and [Gy16b] Gylterud presents a type-theoretic interpretation of a multiset theory, where multisets and propositions are interpreted as types and equality is interpreted by the identity type.

Here we explore the first two degrees of freedom while interpreting set-theoretic equality as a bisimulation relation as in Aczel's original interpretation. Focusing on hset and hprop separately allows us to gain a finer understanding of the original interpretation and to elucidate the role of some of the ideas used in the HoTT book and Gylterud's interpretation.

Overview of the results

Let us consider the type theory given by Martin-Löf type theory augmented with one single W-type $V' := (Wx : hSet_U)\pi_1(x)$, function extensionality and propositional truncation. I will present two main results:

(i) the construction of two interpretations of CZF in this type theory, where the first one interprets sets as hsets and every axiom of CZF is valid. The second interpretation, instead, interprets sets as hsets and formulas as hpropositions using propositional truncations, this interpretation does not validate the Collection axioms;

(ii) a proof-theoretic characterisation of the formulas of CZF validated in the sets-as-hsets interpretation, following [RT06]. The result shows that CZF is not able to distinguish between the model V and the model built with hsets V', so in a sense, all the essential set theoretic information is already captured by the new model V'. This result generalises to all fixed homotopy levels n and therefore gives a hierarchy of models of CZF in type theory indexed by $n \in \mathbb{N}$ that approximate Aczel's model.

References

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