A COINDUCTIVE APPROACH TO TYPE VALUED EQUIVALENCE RELATIONS

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ABSTRACT. We propose a coinductive definition of ∞-equivalence relations in Homotopy Type Theory, where the challenge is of course that one can only ever define structures that are homotopy invariant. In the special case of reflexive relations over the unit type, which are just pointed types, our proposed definition also defines the structure a pointed type may possess in order to be a loop space.

1. INTRODUCTION

Ordinary equivalence relations on a type A are specified as Prop-valued binary relations which are reflexive, symmetric, and transitive. When $\mathcal{R}$ is an equivalence relation in this sense, then the quotient $A/\mathcal{R}$ can be specified as the image of $\mathcal{R} : A \to (A \to \mathsf{Prop})$, as a subtype of $A \to \mathsf{Prop}$, which is always a set. Such 'set-quotients' have been first constructed in [9, 10] using propositional resizing, and as higher inductive types in [8] in the case where A is assumed to be a set.

The next level up consists of '1-equivalence relations', or 'pre-groupoid structures' on a type A, the data of which endows A with the structure of a pre-groupoid in the sense of Ahrens, Kapulkin, Shulman [2]. In the case of pre-groupoid structures one also needs to account for associativity, unit laws, and inverse laws. Groupoid quotients have been defined as higher inductive types in homotopy type theory by Sojakova [7]

In order to formulate a notion of 2-equivalence relations, one requires higher-dimensional structure analogous to that found in the notion of a bi-groupoid. This leads to a combinatorial explosion of data to be specified as one goes higher up the hierarchy of general 'n-equivalence relations'. In this work, we propose a way to bypass the problem of having to specify an infinite amount of coherence data by using a coinductive definition of ∞-equivalence relations involving higher inductive types.

2. MOTIVATING EXAMPLE

Given a function $f : A \to X$, we define the **pre-kernel** $k(f)$ of f to be the reflexive (type-valued) relation on A consisting of

$$k(f)(x,y) :\equiv f(x) = f(y)$$
$$\rho_{k(f)}(x) :\equiv \mathsf{refl}_{f(x)}$$

We can form the reflexive graph quotient of $\langle A, k(f) \rangle$ by forming the pushout

In the join construction [5] this process is iterated, and it is shown that the sequence of join powers f^{*n} of f converges to the image inclusion $\mathsf{im}(f) \hookrightarrow X$ of f. Moreover, the image of f is a model for the ∞-quotient $A/f(x) = f(y)$, because the identification type of $\mathsf{im}(f)$ coincides with that of X.

Now we observe that since $(f * f) \circ \mathsf{inl} = f$, the reflexive relation $f(x) = f(y)$ is compatible with the reflexive relation $(f * f)(x) = (f * f)(y)$, in the sense that the canonical square

$$
\begin{array}{ccc}
\sum_{(x,y:A)} f(x) = f(y) & \dashrightarrow & \sum_{(x,y:A *_X A)} (f * f)(x) = (f * f)(y) \\
\downarrow & & \downarrow \\
A \times A & \xrightarrow[\mathsf{inl} \times \mathsf{inl}]{} & (A *_X A) \times (A *_X A)
\end{array}
$$

is a pullback square.

We therefore found ourselves in the situation that the pre-kernel of a map is a reflexive relation on the domain, which extends to a new reflexive relation on the reflexive graph quotient of the prekernel, which is again a pre-kernel. The idea is now to define an ∞-equivalence relation on A to be a reflexive relation which extends coinductively to a new ∞-equivalence relation on its own reflexive graph quotient.

3. Hereditarily reflexive relations

Since we consider type-valued reflexive relations, we will use the reflexive graph quotients from [6]. Recall that for any type A we have the discrete reflexive graph $\Delta(A)$ of which the type of vertices is A, and the edges are given by the identity type on A. For any reflexive graph $\langle A, R, \rho \rangle$, there is a reflexive graph morphism $\epsilon : \Delta(A) \to \langle A, R, \rho \rangle$, which is the counit of the adjunction $\Delta \dashv \Gamma$, where Γ is the global sections functor of reflexive graphs (given by $\langle A, R, \rho \rangle \mapsto A$).

Given two reflexive graphs $\langle A, R, \rho \rangle$ and $\langle B, S, \sigma \rangle$ and a morphism $f : \langle A, R, \rho \rangle \to \Delta(B)$ a map $f : A \to B$, we can therefore define the composite morphism

$$
\langle A, R, \rho \rangle \xrightarrow{f} \Delta(B) \xrightarrow{\epsilon} \langle B, S, \sigma \rangle
$$

Definition 3.1. Given a map $f : A \to B$, and reflexive relations $\mathcal{R} \equiv \langle R, \rho \rangle$ and $\mathcal{S} \equiv \langle S, \sigma \rangle$ on A and B, respectively, we say that $\mathcal{S}$ **extends** $\mathcal{R}$ along f if either of the following equivalent conditions holds:

(i) The action on edges

$$
(\epsilon \circ f)_1 : \prod_{(x,y:A)} R(x,y) \to S(f_0(x), f_0(y))
$$

of the reflexive graph morphism $\epsilon \circ f : \langle A, R, \rho \rangle \to \langle B, S, \sigma \rangle$ is a fiberwise equivalence.

(ii) The square

$$
\begin{array}{ccc}
\sum_{(x,y:A)} R(x,y) & \dashrightarrow & \sum_{(x,y:B)} S(x,y) \\
\downarrow & & \downarrow \\
A \times A & \xrightarrow[f_0 \times f_0]{} & B \times B
\end{array}
$$

is a pullback square.

Lemma 3.2. *The type of extensions of $\langle R, \rho \rangle$ along $f : (A, R, \rho) \to \Delta(B)$ is equivalent to the fiber of the map* $\mathsf{rRel}(f_0) : \mathsf{rRel}(B) \to \mathsf{rRel}(A)$ *at $\langle R, \rho \rangle$, where this map is just defined by substitution.*

The following is a coinductive definition that can be made precise using indexed containers [3, 1].

Definition 3.3. We say that a reflexive relation $\langle R, \rho \rangle$ on a type A is **hereditarily reflexive** if there is a reflexive relation $\langle S, \sigma \rangle$ on $\mathsf{colim}(A, R, \rho)$ which extends $\langle R, \rho \rangle$ along $\mathsf{constr}(A, R, \rho) : (A, R, \rho) \to \Delta(\mathsf{colim}(A, R, \rho))$ such that $\langle S, \sigma \rangle$ is again hereditarily reflexive.

Example 3.4. The prekernel $k(f)$ of a map f can be given the structure of a hereditarily reflexive relation. This defines the **kernel** $\mathcal{K}(f)$ of a map, which consists of the prekernel $k(f)$ together with its canonical structure of a hereditarily reflexive relation.

Given a hereditarily reflexive relation $\mathcal{R}$ on A, we can construct a sequence of reflexive graphs

$$
(A_0, R_0, \rho_0) \xrightarrow{\epsilon \circ \mathsf{constr}} (A_1, R_1, \rho_1) \xrightarrow{\epsilon \circ \mathsf{constr}} (A_2, R_2, \rho_2) \xrightarrow{\epsilon \circ \mathsf{constr}} \cdots
$$

and one can define the **hereditarily reflexive quotient** $A/\mathcal{R}$ to be the sequential colimit of the underlying sequence of types. The quotient map $q_{\mathcal{R}} : A \to A/\mathcal{R}$ is obtained from the cone structure on

$A/\mathcal{R}$. One can show that the quotient map $q_\mathcal{R}$ is surjective for any hereditarily reflexive relation $\mathcal{R}$. This defines an operation

$$\mathcal{Q} : \left(\textstyle\sum_{(\langle R,\rho \rangle : \mathsf{rRel}(A))} \mathsf{isHRR}(R,\rho) \right) \to \textstyle\sum_{(B:\mathsf{U})} \textstyle\sum_{(q:A \to B)} \mathsf{isSurj}(q).$$

Our goal is now to show that $\mathcal{Q}$ is a homotopy inverse to the kernel operation $\mathcal{K}$ restricted to the surjective maps.

The fact that $\mathcal{Q} \circ \mathcal{K} \sim \mathsf{id}$ follows from the following theorem.

Theorem 3.5. *Given a map $f : A \to X$, we have a commuting triangle*

in which the bottom map is an equivalence.

The following theorem asserts that the prekernel of the quotient map agrees with the underlying reflexive relation of a hereditarily reflexive relation. This is a partial result towards showing that $\mathcal{K} \circ \mathcal{Q} = \mathsf{id}$. Note that this equality states precisely that the quotienting operation $\mathcal{Q}$ is *effective*.

Theorem 3.6. *Let $\mathcal{R}$ be a hereditarily reflexive relation with underlying relation $R : A \to A \to \mathsf{U}$. Then there is a canonical equivalence*

$$(q_\mathcal{R}(x) = q_\mathcal{R}(y)) \simeq R(x,y).$$

It remains to extend this equality of reflexive relations to an equality of hereditarily reflexive relations. This is work in progress. We are in the process of formalizing our work in the Coq proof assistant using the HoTT library [4].

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