Differential Cohesive Type Theory (Extended Abstract)*

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As internal languages of toposes, type theories allow mathematicians to reason *synthetically* about mathematical structures in a concise, natural, and computer-checkable way. The basic system of homotopy type theory (Martin-Löf type theory extended with higher inductive types and Voevodsky's univalence axiom) provides an internal language for $(\infty,1)$-toposes, and has enabled a significant line of work on synthetic homotopy theory. However, some mathematics of interest is difficult to formulate synthetically in this basic setting, which has prompted the investigation of extensions of homotopy type theory to describe $(\infty,1)$-toposes with extra structure. For example, real-cohesive homotopy type theory (Shulman, 2015) equips each type with a synthetic notion of topology, in addition to the homotopy structure given by the path type. Operations on these topologies are expressed by three modalities $\flat A$ (retopologize the type A discretely), $\sharp A$ (retopologize the type A codiscretely), and $\int A$ (make topological paths into homotopical paths), which form an adjoint triple $\int \dashv \flat \dashv \sharp$. These modalities allow synthetic homotopy theory to be used to prove topological statements, e.g. proving the Brouwer fixed-point theorem—that all continuous maps over the *topological* disk have a fixed point—using the synthetic proof of $\pi_1(S^1) = \mathbb{Z}$ for the *homotopical* (higher-inductive) circle.

The concept of cohesive toposes can be extended to *differential cohesive toposes*, which express extra structure that is of interest in algebraic and differential geometry. Examples of differential cohesive toposes include smooth manifolds and their ∞-stack variants, which are of great interest in current pure mathematics. For example, Sati et al. (2012) use spaces locally modeled on 2- and 6-types. Furthermore, differential cohesive toposes are used to great extent by Schreiber (2013) to reason about spaces with geometric structures of interest to modern physics, and the differential part of differential cohesion plays an important part in algebraic geometry, as first noted in the form of an adjoint triple by Simpson and Teleman (1997). In addition to the cohesive modalities $\int$, $\flat$, and $\sharp$, differential cohesion adds three more modalities, $\Re$, $\Im$, and $\&$, which axiomatize an abstract notion of *infinitesimal direction* that is used to access the differential structure. For example, $\Re A$ is the underlying space of A without infinitesimal directions, while $\Im A$ is the underlying space of A but with all maps made trivial on tangent spaces. $\Re$ and $\&$ are comonadic modalities, while $\Im$ is a monadic modality; they form an adjoint triple $\Re \dashv \Im \dashv \&$; and they have certain relationships with the cohesive modalities (e.g. $\int$ is contained in $\Im$ and $\flat$ is contained in $\&$). Initial work by Wellen (2017) postulates a monadic modality $\Im$ in standard homotopy type theory and develops the basics of Cartan geometry, which suggests that enriching homotopy type theory with the differential cohesion modalities could be a useful tool for working synthetically in this setting.

In this work, we begin an investigation of extensions of type theory to express differential cohesion. A central challenge is to design a type theory with all six type operators: the cohesive modalities $\int$, $\flat$, and $\sharp$, as well as the differential modalities $\Re$, $\Im$, and $\&$, all with the desired properties and relationships to each other. Because $\flat$ and $\Re$ and $\&$ are comonadic modalities, which unlike monadic modalities cannot be postulated (Shulman, 2015), this design must involve changing the judgmental structure of the type theory. In our talk, we can report on a simply-typed λ-calculus that describes the structure of the six modalities as functors on closed types/contexts, which we have designed using recent work on adjoint type theory (Licata

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et al., 2017). We can also describe work-in-progress to express the fiberwise behavior of these modalities on dependent types.

Contributions. Real-cohesive homotopy type theory (Shulman, 2015) includes a subsystem *spatial type theory* with only the $\flat$ and $\sharp$ modalities. In this subsystem, the $\flat$ comonadic modality is represented by rules similar to the judgmental presentation of the necessitation comodality in modal logic (Pfenning and Davies, 2001). A special kind of *crisp* variable is used to define the $\flat$ type, while ordinary variables are called *cohesive*; a typing judgment $\Gamma \mid \Delta \vdash e : \tau$ is continuous on the cohesive variables in Δ, but may be discontinuous on the crisp variables in Γ. While the $\flat$ and $\sharp$ modalities are basic type constructors, the $\int$ (fundamental ∞-groupoid, turning topological paths into homotopy paths) is defined as a higher inductive type, a localization (nullification) with respect to the Dedekind-reals. This approach admits constructions of familiar topological spaces using the real numbers, including certain topological spheres or disks.

However, in differential cohesion, it is not clear that $\int$ should be axiomatized as a localization (in typical models of differential cohesion, the Dedekind reals do not correspond to the real line as a smooth space), so our first contribution is to formulate an abstract shape modality. We have given a *mode theory* in adjoint logic (Licata et al., 2017) that expresses all three cohesive modalities, and because cut elimination/normalization holds for all mode theories in adjoint logic, we obtain it as a corollary. We have also adapted the mode theory to a simpler presentation by specializing the rules of adjoint logic to this particular mode theory, resulting in a typing judgment with three sorts of contexts, written $\Gamma \mid \Delta \mid \Xi \vdash e : \tau$. Here, Γ and Δ still hold crisp and cohesive variables, respectively, and where Ξ contains *shapely* variables, which are constant on the connected components of the topological structure. Crisp variables correspond to the modality $\flat$, while shapely variables correspond to the modality $\int$.

Our second contribution is to extend this to the differential cohesive modalities. We have given a mode theory in adjoint type theory for this and again unpacked a nice formulation of the specialized rules. The result is a judgment $\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e : \tau$ that uses five different sorts of contexts, corresponding to the modalities $\flat$, $\Re$, identity, $\Im$, and $\int$ respectively. The rules for the cohesive and differential cohesive types manipulate these contexts in subtle ways, which express the relationships between the modalities.

In work in progress, we are extending this type theory to dependent types, which is not straightforward because of the dependency structure of the differential cohesive modalities. In spatial type theory (Shulman, 2015), cohesive variables can depend on crisp variables but not vice versa; but in our reformulation both crisp and cohesive variables will also depend on shapely variables, and vice versa. Accounting for this will require a more nuanced view on the dependency graph of contexts and types.

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