# Fibred Fibration Categories

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## Fibred type-theoretic fibration categories

A "good" notion of fibred category between categorical models of dependent type theory.

- Construct total type-theoretic structure from fiberwise one
- Change of base

#### **Application**

Categorical description of "logical relations" [Hermida, 1993] on HoTT.

#### **Theorem**

 $t: \Pi_{A:\mathcal{U}}\Pi_{x:A}x = x \to x = x$  is homotopic to  $\lambda(p: x = x).p^n$  for some  $n \in \mathbb{Z}$ .

# Type-theoretic fibration categories

Type-theoretic fibration categories [Shulman, 2015] are sound and complete categorical models of Martin-Löf's intentional theory.

- ► A category C equipped with specific morphisms called *fibrations* corresponding to type families.
- Path induction is modeled by the lifting property.

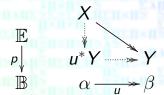


A morphism like refl is called an acyclic cofibration.

#### Fibred type-theoretic fibration categories

A fibred type-theoretic fibration category is a fibred category  $p : \mathbb{E} \to \mathbb{B}$  such that:

- 1.  $\mathbb{E}$  and  $\mathbb{B}$  are type-theoretic fibration categories and p preserves all structures of type-theoretic fibration category.
- 2. A fibration in  $\mathbb{E}$  factors as a *vertical* fibration followed by a *horizontal* fibration.



3. Every Cartesian morphism above an acyclic cofibration is an acyclic cofibration.

#### Proposition

A fibred category  $p:\mathbb{E}\to\mathbb{B}$  is a fibred type-theoretic fibration category if and only if:

- 1.  $\mathbb B$  and all fibers  $\mathbb E_{\alpha}$  are type-theoretic fibration categories.
- 2. For any morphism  $u : \alpha \to \beta$  in  $\mathbb{B}$ ,  $u^* : \mathbb{E}_{\beta} \to \mathbb{E}_{\alpha}$  is a strong fibration functor.
- 3. For every acyclic cofibration  $u: \alpha \mapsto \beta$  and fibration  $f: Y \twoheadrightarrow X$  in  $\mathbb{E}_{\beta}$ ,  $u^*: \mathbb{E}_{\beta}/X(X,Y) \to \mathbb{E}_{\alpha}/u^*X(u^*X,u^*Y)$  is surjective.
- 4. For every fibration  $u : \alpha \rightarrow \beta$  in  $\mathbb{B}$ ,  $u^* : \mathbb{E}_{\beta} \rightarrow \mathbb{E}_{\alpha}$  has a right adjoint  $u_*$  satisfying the "Beck-Chevalley condition".

A fully categorical proof is in arXiv:1602.08206. I give a syntactic description.

A fibred category  $p : \mathbb{E} \to \mathbb{B}$  models a type theory with two sorts Kind and Type.

Category theory	Type theory
$\alpha \in \mathbb{B}$	$\alpha$ : Kind
$X \in \mathbb{E}_{lpha}$	$X: \alpha \to Type$

The Proposition says the pairs of  $(\alpha: \mathsf{Kind}, X: \alpha \to \mathsf{Type})$  form a new type theory.

Concepts	Definition	
type	$(\alpha : Kind, X : \alpha \to Type)$	
element	$(a:\alpha,x:X(a))$	
family	$(\beta: \alpha \to Kind,$	
	$Y:\Pi_{a:lpha}eta(a) o X(a) o Type)$	
section	$(u:\Pi_{a:\alpha}\beta(a),$	
	$f: \Pi_{a:\alpha}\Pi_{x:X(a)}Y(a,u(a),x))$	
pair	$((a,b): \sum_{a:\alpha} \hat{\beta}(a),$	
	$(x,y): \Sigma_{x:X(a)}Y(a,b,x)$	
identity type ?		

We need the path induction on an identity kind w.r.t. any type family over the identity kind.

$$lpha$$
: Kind  $X:\Pi_{a,a':lpha}a=a' o \mathsf{Type}$   $x:\Pi_{a:lpha}X(a,a,\mathsf{refl}_a)$   $\mathsf{ind}_{=_lpha}(X,x):\Pi_{a,a':lpha}\Pi_{p:a=a'}X(a,a',p)$   $\mathsf{ind}_{=_lpha}(X,x,a,a,\mathsf{refl}_a)\equiv x$ 

In particular, for  $\alpha$ : Kind,  $X: \alpha \to \text{Type}$  and  $p: a =_{\alpha} a'$ , we have the transport along p  $p_*: X(a) \to X(a')$ .

The identity type of  $(\alpha: Kind, X: \alpha \rightarrow Type)$  is the pair of

- $\blacksquare$  =:  $\alpha \to \alpha \to \mathsf{Kind}$  and
- $lacksquare \lambda_a a' pxx'. p_* x = x' : \Pi_{a,a':lpha} \Pi_{p:a=a'} X(a) 
  ightarrow X(a') 
  ightarrow \mathsf{Type}.$

It is the type of "path over path" but in different sorts.

### Universes in a fibred setting

Let  $\mathcal{U}$ : Kind and  $\mathcal{V}$ : Type be universes of kinds and types. Then  $(\mathcal{U}, \lambda(\alpha : \mathcal{U}).\alpha \to \mathcal{V})$  is a universe in the new type theory if, for any  $\alpha : \mathcal{U}$  and  $X : \alpha \to \mathcal{V}$ ,  $\Pi_{a:\alpha}.X(a) : \mathcal{V}$ . Its elements are  $(\alpha : \mathcal{U}, X : \alpha \to \mathcal{V})$ .

# Equivalences in a fibred setting

For  $(u: \alpha \to \beta, f: \Pi_{a:\alpha}X(a) \to Y(u(a)))$ , An element of is-equiv(u, f) is (v: homotopy inverse of <math>u, g: homotopy inverse of f above <math>v).

#### Lemma

Suppose the function extensionality holds. Then

$$\mathsf{is\text{-}equiv}(u,f) \simeq (\mathsf{is\text{-}equiv}(u), \lambda\_. \Pi_{a:\alpha} \mathsf{is\text{-}equiv}(f_a))$$

for all  $(u: \alpha \to \beta, f: \Pi_{a:\alpha}X(a) \to Y(u(a)))$  in the new type theory.

#### Univalence in a fibred setting

A universe U in a type theory is *univalent* if the canonical map

 $\lambda(A:U).(A,A,\mathsf{id}_A):U\to \Sigma_{A,A':U}A\simeq A'$  is an equivalence.

The new universe  $(\mathcal{U}, \lambda(\alpha : \mathcal{U}).\alpha \to \mathcal{V})$  is univalent if  $\lambda(\alpha : \mathcal{U}).(\alpha, \alpha, \mathrm{id}_{\alpha})$  is an equivalence and for all  $\alpha : \mathcal{U}, \lambda X.(X, X, \lambda(a : \alpha).\mathrm{id}_{X(a)}) : (\alpha \to \mathcal{V}) \to \Sigma_{X,Y:\alpha\to\mathcal{V}}\Pi_{a:\alpha}X(a) \simeq Y(a)$  is an equivalence. This holds if  $\mathcal{U}$  and  $\mathcal{V}$  are univalent and the function extensionality holds.

# Outline Examples

#### Arrow categories

Let  $\mathbb{C}$  be a type-theoretic fibration category and write  $(\mathbb{C}^{\to})_f \subset \mathbb{C}^{\to}$  for the full subcategory of all the fibrations. Then  $\operatorname{\mathbf{cod}}:(\mathbb{C}^{\to})_f \to \mathbb{C}$  is a fibred type-theoretic fibration category. In this case Type  $\equiv$  Kind.

If  $\mathbb{C}$  has a univalent universe, so does  $(\mathbb{C}^{\to})_f$ . Originally, Shulman proved  $(\mathbb{C}^{\to})_f$  is a type-theoretic fibration category [Shulman, 2015], and I give a fibred categorical description.

#### Change of base

Let  $p: \mathbb{E} \to \mathbb{B}$  be a fibred type-theoretic fibration category and  $F: \mathbb{A} \to \mathbb{B}$  be a functor preserving fibrations, pullbacks of fibrations and acyclic cofibrations. Then the change of base or pullback

$$F^*\mathbb{E} \longrightarrow \mathbb{E}$$

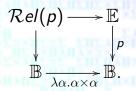
$$\downarrow \qquad \qquad \downarrow p$$

$$\mathbb{A} \xrightarrow{F} \mathbb{B}$$

is a fibred type-theoretic fibration category.

#### Relational model

Let  $p:\mathbb{E}\to\mathbb{B}$  be a fibred type-theoretic fibration category. We have the change of base



 $\mathcal{R}el(p)$  is the category of binary families  $(\alpha: \mathsf{Kind}, R: \alpha \to \alpha \to \mathsf{Type}).$ 

Let  $t: \Pi_{A:\mathcal{U}}\Pi_{x:A}x = x \to x = x$ .

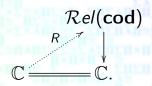
#### **Theorem**

▶ t is "natural": for any  $f:A \to B$  in  $\mathcal{U}$ ,  $\mathsf{ap}_f \circ t \sim t \circ \mathsf{ap}_f$ .

$$\begin{array}{c}
x = x \xrightarrow{t} x = x \\
\operatorname{ap}_{f} \downarrow & \operatorname{\downarrow} \operatorname{ap}_{f} \\
fx = fx \xrightarrow{t} fx = fx
\end{array}$$

If the type theory has  $\mathbb{S}^1$ , then for some  $n \in \mathbb{Z}$ ,  $tp = p^n$  for all p : x = x.

Let  $\mathbb C$  be the syntactic category of Martin-Löf type theory with a univalent universe  $\mathcal U$ . Then  $(\mathcal U, \lambda(A, B:\mathcal U).A \to B \to \mathcal U)$  is a univalent universe in  $\mathcal Rel(\mathbf{cod})$ . We have a strong fibration functor



In particular, for every closed term t: A, we have  $R_A: A \to A \to \mathsf{Type}$  and  $R_t: R_A(t,t)$ .

Let  $t: \Pi_{A:\mathcal{U}}\Pi_{x:A}x = x \to x = x$ . Then  $R_t: \Pi_{A,B:\mathcal{U},W:A\to B\to \mathcal{U}}\Pi_{x:A,y:B,v:W(x,y)}\Pi_{p:x=x,q:y=y}(p,q)_*v = v \to (tp,tq)_*v = v$ .

$$\begin{array}{cccc}
x & \xrightarrow{p} & x & x & \xrightarrow{tp} & x \\
v & = & |v| & \rightarrow & v & = & |v| \\
y & \xrightarrow{q} & y & y & \xrightarrow{tq} & y
\end{array}$$

Given  $f: A \to B$  in  $\mathcal{U}$ , let  $W(x, y) \equiv fx = y$ . Then  $R_t$  looks like

Let  $y \equiv fx$ ,  $q \equiv \operatorname{ap}_f p$  and  $v \equiv \operatorname{refl}_{fx}$  and apply the function to  $\operatorname{refl}_{\operatorname{ap}_f p}$ . Then  $\operatorname{ap}_f(tp) = t(\operatorname{ap}_f p)$  for all x : A and p : x = x.

Suppose the type theory has  $\mathbb{S}^1:\mathcal{U}$  with  $b:\mathbb{S}^1$  and l:b=b. Then  $(\mathbb{S}^1,=_{\mathbb{S}^1})$  is a unit circle in  $\mathcal{R}el(\mathbf{cod})$  and we still have the functor  $R:\mathbb{C}\to\mathcal{R}el(\mathbf{cod})$ . Let y:B and q:y=y which corresponds to  $f:\mathbb{S}^1\to B$ .  $t(l)=l^n$  for some  $n\in\mathbb{Z}$  and

$$tq = t(ap_f(I))$$

$$= ap_f(t(I))$$

$$= ap_f(I^n)$$

$$= (ap_f(I))^n$$

$$= q^n$$

#### References I



Shulman, M. (2015).

Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*,
25(05):1203–1277.