Modelling Cubical Type Theory in Agda

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Workshop on HoTT/UF '16, Porto

Overview

This material in this talk is based on our paper:

Axioms for Modelling Cubical
Type Theory in a Topos, CSL 2016

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- Our axiomatisation

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Covering:

- The internal type theory of a topos
- Translating this into Agda
- Our axiomatisation
- Why this is a good approach

The internal type theory of a topos

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- Standard interpretation of extensional MLTT in a category with families (CwF) associated with any topos $\mathcal{E}$ (with families over $X \simeq \mathcal{E}/X$).

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- The subobject classifier Ω becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
- The universal property of Ω gives rise to comprehension subtypes...

Comprehension subtypes

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\frac{\Gamma, x : A \vdash \varphi(x) : \Omega}{\Gamma \vdash \{x : A \mid \varphi(x)\}}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.

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- Function extensionality

An impredicative universe of propositions

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```
{-# OPTIONS --type-in-type #-}
-- the following definition relies on type-in-type,
-- which is switched on only in this module

record Ω : Set where
  constructor prop
  field
    prf : Set
    equ : (u v : prf) → u ≡ v
```

1. www.cs.bham.ac.uk/~mhe/impredicativity/

Comprehension subtypes

We simply form the sigma type:

```
----------------------------------------
-- Comprehension
----------------------------------------
set : (A : Set)(P : A → Ω) → Set
set A P = Σ x ∈ A , prf (P x)

syntax set A (λ x → P) = ⟦ x ∈ A | P ⟧
```

Comprehension subtypes

For example:

```
Evens : Set
Evens = ⟦ n ∈ ℕ ∣ ∃ m ∈ ℕ , 2 * m ≈ n ⟧

four : Evens
four = (4 , ∣ 2 , refl ∣)
```

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Internalisation of the face lattice $\mathbb{F}$. Think of elements of $\mathtt{Cof}$ as propositions such as $(i = 1)$, $(i = 0) \vee (j = 1)$, $(i = 0) \wedge (j = 0)$ etc.

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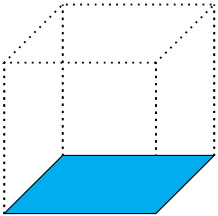

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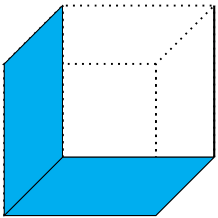

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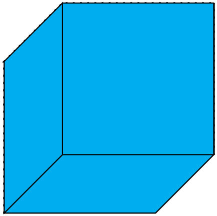

Modelling partial terms/types

How do we model partial types?

$$\Gamma, \varphi \vdash A$$

And partial terms?

$$\Gamma, \varphi \vdash a : A$$

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We will make extensive use of these types in connection with partial elements.

Partial elements

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- $f : [\varphi] \to A$.

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We say that a partial element (φ, f) extends to $a : A$ if the following relation holds:

$$(\varphi, f) \nearrow a \triangleq \forall (u : [\varphi]).\ f\, u = a$$

Filling in the internal TT

The notion of Kan filling in our internal type theory:

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The type of filling structures for $\mathtt{I}$-indexed families of types, $\mathtt{Fill} : (e : \{\mathtt{0}, \mathtt{1}\})(A : \mathtt{I} \to \mathcal{U}) \to \mathcal{U}$, is defined by

$$
\begin{aligned}
&\mathtt{Fill}\, e\, A \triangleq \\
&\qquad (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A) \\
&\qquad (a : \{a' : A\, e \mid (\varphi, f) @ e \nearrow a'\}) \\
&\qquad \to \\
&\qquad \{g : \Pi_{\mathtt{I}} A \mid (\varphi, f) \nearrow g \ \wedge \ g\, e = a\}
\end{aligned}
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Filling in Agda

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\end{aligned}
$$

```
Fill e A =
  (ϕ : Cof)(f : [ ϕ ] → Π A)
  (a : ⟦ a' ∈ A ⟨ e ⟩ ∣ (ϕ , f) • ⟨ e ⟩ ↗ a' ⟧)
  → -------------------------------------------
  ⟦ g ∈ Π A ∣ ((ϕ , f ) ↗ g) & (g ⟨ e ⟩ ≈ fst a) ⟧
```

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  – easily translate them into Agda

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- Construct proofs internally
  – easily translate them into Agda
- Look for models of these axioms
  – e.g. classifying topos for the theory

Thanks for listening!

Axioms for Modelling Cubical Type Theory in a Topos

Ian Orton and Andrew Pitts, CSL 2016

Paper and Agda:
http://www.cl.cam.ac.uk/~rio22/

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