Univalence

FOR

DUMMIES

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- Email discussion with David McAllester
- Why do we believe that univalence is sound?
- Doesn’t understand the simplicial set model.

The Simplicial Model of Univalent Foundations
Chris Kapulkin, Peter LeFanu Lumsdaine, Vladimir Voevodsky

Groupoids!

July 3, 2015, Friday
TLCA Invited Talk
(chair: Peter Dybjer)
9:00: Martin Hofmann.
The Groupoid Interpretation of Type Theory, a Personal Retrospective

- Groupoids : univalent universe of sets
- Setoids : univalent universe of propositions
- Can we do 2-groupoids?
- This gets complicated !

Type Theory eats itself without indigestion

joint work with Ambrus Kaposi

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Type Theory in Type Theory?

Type Theory should eat itself
James Chapman, LFMTP 2008

A Formalisation of a
Dependently Typed Language
as an Inductive-Recursive Family
Nils Anders Danielsson
TYPES 2006

- No pre-terms ! Only typed objects.
- Verified Metatheory
- Template Type Theory

Guilhem Jaber, Nicolas Tabareau,
Matthieu Sozeau.
Extending Type Theory with Forcing.
LICS 2012

```
data Ty : Set where
  ι    : Ty
  _⇒_ : Ty → Ty → Ty

data Con : Set where
  •   : Con
  _,_ : Con → Ty → Con

data Var : Con → Ty → Set where
  zero : Var (Γ , σ) σ
  suc  : Var Γ σ → Var (Γ , τ) σ

data Tm :  Con → Ty → Set where
  var : Var Γ σ → Tm Γ σ
  _$_ : Tm Γ (σ ⇒ τ) → Tm Γ σ → Tm Γ τ
  ƛ   : Tm (Γ , σ) τ → Tm Γ (σ ⇒ τ)
```

Simply Typed λ-calculus

OMITTED

- Substitution

```
_[_] : Tm Γ σ → Tms Γ Δ → Tm Γ σ
```

- β η - Equality

```
data _~_ : Tm Γ σ → Tm Γ σ → Set
```

- Terms as quotient

```
Tm Γ σ / ~
```

Dependent Types

```
data Con : Set
data Ty : Con → Set
data Tm : (Γ : Con) → Ty Γ → Set
data Tms : Con → Con → Set
```

Induction-Induction

```
_,_   : (Γ : Con) → Ty Γ → Con
```

```
_[_]T : Ty Δ → Tms Γ Δ → Ty Γ
```

```
_,_   : (δ : Tms Γ Δ){A : Ty Δ} → Tm Γ (A [ δ ]T) → Tms Γ (Δ , A)
```

A categorical semantics for inductive-inductive definitions
TA, Frederik Forsberg, Peter Morris and Anton Setzer
CALCO 2011

Coerce

```
coe   : A ~Ty B → Tm Γ A → Tm Γ B
```

Dependent Types II

```
data Con : Set
data Ty : Con → Set
data Tm : (Γ : Con) → Ty Γ → Set
data Tms : Con → Con → Set
data _~Con_ : Con → Con → Set
data _~Ty_ : Ty Γ → Ty Γ → Set
data _~Tm_ : Tm Γ A → Tm Γ A → Set
data _~Tms_ : Tms Γ Δ → Tms Γ Δ → Set
```

Boilerplate

- ~s are equivalence relations
- constructors are congruences
- Ty, Tm, Tms are families of setoids


This way I will never finish my Type Theory in Type Theory

Higher Inductive Types (HITs) to the rescue

```
data S¹ : Set where
  base : S¹
  loop : base ≡ base
```

- HITs which are sets can be useful.
- Quotient Inductive Types (QITs)
- Examples in the HoTT book:
  - Cauchy reals (11.3)
  - Cumulative hierarchy of sets (10.5)

The infinite tree example

```
data T₀ : Set where
  leaf : T₀
  node : (ℕ → T₀) → T₀
```

```
data _~_ : T₀ → T₀ → Set where
  leaf : leaf ~ leaf
  node : (∀ {n} → f n ~ g n) → node f ~ node g
  perm : isIso f → node g ~ node (g ∘ f)
```

```
T = T₀ / _~_
```

Define !

```
nodeT : (ℕ → T) → T
```

```
[ node f ] ≡ nodeT (λ i → [ f i ])
```

Infinite trees as a QIT

```
data T : Set where
  leaf : T
  node : (ℕ → T) → T
  perm : isIso f → node g ≡ node (g ∘ f)
  isSet  : {e0 e1 : u ≡ v} → e0 ≡ e1
```

Dependent types as a QIIT

```
data Con where
  •      : Con
  _,_    : (Γ : Con) → Ty Γ → Con

data Ty where
  _[_]T : Ty Δ → Tms Γ Δ → Ty Γ
  U      : Ty Γ
  El     : (A : Tm Γ U) → Ty Γ
  Π      : (A : Ty Γ)(B : Ty (Γ , A)) → Ty Γ

data Tms where
  ε    : Tms Γ •
  _,_  : (δ : Tms Γ Δ) → Tm Γ (A [ δ ]T) → Tms Γ (Δ , A)
  id   : Tms Γ Γ
  _∘_   : Tms Δ Σ → Tms Γ Δ → Tms Γ Σ
  π₁    : Tms Γ (Δ , A) →  Tms Γ Δ

data Tm where
  _[_]t : Tm Δ A → (δ : Tms Γ Δ) → Tm Γ (A [ δ ]T)
  π₂     : (δ : Tms Γ (Δ , A)) → Tm Γ (A [ π₁ δ ]T)
  app    : Tm Γ (Π A B) → Tm (Γ , A) B
  lam    : Tm (Γ , A) B → Tm Γ (Π A B)
```

```
[id]T : A [ id ]T ≡ A
[][]T : (A [ δ ]T) [ σ ]T ≡ A [ δ ∘ σ ]T
U[]   : U [ δ ]T ≡ U
El[]  : El A [ δ ]T ≡ El (coe (TmΓ= U[]) (A [ δ ]t))
Π[]   : (Π A B) [ δ ]T ≡ Π (A [ δ ]T) (B [ δ ^ A ]T)
```

```
idl    : id ∘ δ ≡ δ
idr    : δ ∘ id ≡ δ
ass    : (σ ∘ δ) ∘ ν ≡ σ ∘ (δ ∘ ν)
,∘     : (δ , a) ∘ σ ≡ (δ ∘ σ) , coe .. (a [ σ ]t)
π₁β    : π₁ (δ , a) ≡ δ
πη     : (π₁ δ , π₂ δ) ≡ δ
εη     : {σ : Tms Γ •} → σ ≡ ε
```

```
[id]t : t [ id ]t ≡[ [id]T ]≡ t
[][]t : (t [ δ ]t) [ σ ]t ≡[ [][]T ]≡  t [ δ ∘ σ ]t
π₂β   : π₂ (δ , a) ≡[ π₁β ]≡ a
lam[] : (lam t) [ δ ]t ≡[ Π[] ]≡ lam (t [ δ ^ A ]t)
Πβ    : app (lam t) ≡ t
Πη    : lam (app t) ≡ t
```

The Recursor

```
record Motives : Set₁ where
  field
    Conᴹ : Set
    Tyᴹ  : Conᴹ → Set
    Tmsᴹ : Conᴹ → Conᴹ → Set
    Tmᴹ  : (Γᴹ : Conᴹ) → Tyᴹ Γᴹ → Set
```

```
record Methods (M : Motives) : Set₁ where
  field
    •ᴹ     : Conᴹ
    _,Cᴹ_  : (Γᴹ : Conᴹ) → Tyᴹ Γᴹ → Conᴹ
    ...
```

```
module rec (M : Motives)(m : Methods M) where

  Con-elim : Con → Conᴹ
  Ty-elim  : (A : Ty Γ) → Tyᴹ (Con-elim Γ)
  Tms-elim : (δ : Tms Γ Δ)→ Tmsᴹ(Con-elim Γ)(Con-elim Δ)
  Tm-elim  : (t : Tm Γ A) → Tmᴹ (Con-elim Γ) (Ty-elim A)
```

Motives + Methods

=

Algebras

=

Models of TT

Set theoretic model

Problem: Set is not a set!

```
data UU : Set
EL : UU → Set

data UU where
   'Π' : (A : UU) → (EL A → UU) → UU
   'Σ' : (A : UU) → (EL A → UU) → UU
   '⊤' : UU

EL ('Π' A B) =    (x : EL A) → EL (B x)
EL ('Σ' A B) = Σ (EL A) λ x → EL (B x)
EL '⊤' = ⊤
```

```
M : Motives
M = record
      { Conᴹ =                    UU
      ; Tyᴹ  = λ ⟦Γ⟧      → EL ⟦Γ⟧ → UU
      ; Tmsᴹ = λ ⟦Γ⟧ ⟦Δ⟧ → EL ⟦Γ⟧ → EL ⟦Δ⟧
      ; Tmᴹ  = λ ⟦Γ⟧ ⟦A⟧ → (γ : EL ⟦Γ⟧) → EL (⟦A⟧ γ)
      }
```

```
m : Methods M
m = record
      { •ᴹ      =                  '⊤'
      ; _,Cᴹ_   = λ ⟦Γ⟧ ⟦A⟧ → 'Σ' ⟦Γ⟧ ⟦A⟧

      ...

      ; [id]Tᴹ = refl
      ; [][]Tᴹ = refl

      ...
```

```
⟦_⟧C : Con → UU
⟦_⟧T : Ty Γ → EL (⟦ Γ ⟧C) → UU
⟦_⟧s : Tms Γ Δ → EL (⟦ Γ ⟧C) → EL (⟦ Δ ⟧C)
⟦_⟧t : (t : Tm Γ A) → (γ : EL (⟦ Γ ⟧C)) → EL (⟦ A ⟧T γ)
```

The logical predicate translation (almost finished)

- Inspired by JP Bernardy et al on parametricity for dependent types
- A syntactic translation assigning to
  - each context, an extended context
  - to every type, a logical predicate
  - to every term, a proof that the term satisfies the logical predicate.
  - requires dependent eliminator

The presheaf interpretation (started)

- Fix a category C
- Contexts are interpreted as presheaves
- Types as families of presheaves
- Substitutions are natural transformations
- Terms are global sections

Normalisation by evaluation

- Normal forms are a presheaf over the category of variable substitutions.
- We can generalise NBE from simple types to dependent types.
- However, the normal forms have types which are not normal.

Normal forms with normal types?

- Can we define a mutual datatype of normal forms with normal types?
- No equations, no truncation!
- Use this to define semi-simplicial types?

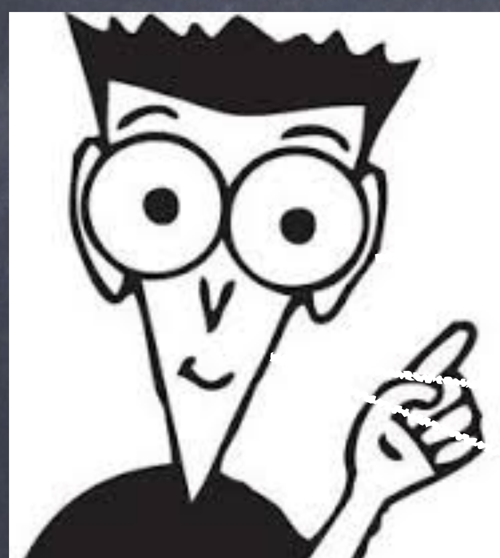
- We need to define normalisation mutual with normal forms!
- Even in the simplest case (only variables) this leads to a new coherence problem!
- substitution is defined by recursion

```
_[_]T : Ty Δ → Vars Γ Δ → Ty Γ
_[_]v : Var Δ A → (δ : Vars Γ Δ) → Var Γ (A [ δ ]T)

data Vars where
  ε      : Vars Γ •
  _,_    : (δ : Vars Γ Δ){A : Ty Δ} → Var Γ (A [ δ ]T)
             → Vars Γ (Δ , A)

wk : {A : Ty Γ} → Vars (Γ , A) Γ

data Var where
  vz : Var (Γ , A) (A [ wk ]T)
  vs : Var Γ A → Var (Γ , B) (A [ wk ]T)
```



Complete Failure !

Your goal was to model
univalent universes
but you can only eliminate
into a set!

2-level theory

- Start with a strict type theory (with K)
- Introduce a universe with a univalent equality, can only eliminate into the universe.
- Syntax of Type Theory has to be defined in the strict theory
- However we can use the univalent universe to build models.