An inductive dependently-typed construction of simplicial sets and of similar presheaves over a Reedy category

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Background: presheaf vs inductive definition of semi-simplicial sets

The presheaf definition:

$$\begin{array}{ll} X_0 & \text{(points)} \\ X_1 & \text{(line segments)} \\ X_2 & \text{(triangles)} \\ \vdots & \\ X_n & (n\text{-simplices}) \\ \vdots & \end{array}$$

equipped with faces $d_i^n : X_n \to X_{n-1}$ satisfying $d_i^n \circ d_j^{n+1} = d_j^n \circ d_{i+1}^{n+1}$ for $n \geq i \geq j \geq 0$

Faces can be hard-wired by considering instead the family

$$\begin{array}{l} Y_0 \\ \Sigma a, b : Y_0.\ Y_1(a,b) \\ \Sigma a, b, c : Y_0.\ \Sigma x : Y_1(a,b). \Sigma y : Y_1(a,c). \Sigma z : Y_1(b,c).\ Y_2(a,b,c,x,y,z) \\ \vdots \end{array}$$

where we have set:

$$\begin{array}{lll} Y_0 & \triangleq X_0 & \\ Y_1(a,b) & \triangleq \{x : X_1 | d_1^1(x) = a, d_0^1(x) = b\} & \text{for } a, b : Y_0 \\ Y_2(a,b,c,x,y,z) & \triangleq \{t : X_2 | d_2^2(t) = x, d_1^2(t) = y, d_0^2(t) = z\} & \text{for } a, b, c : Y_0, x : Y_1(a,b) \\ & & \quad y : Y_1(a,c), z : Y_1(b,c) \\ \vdots & & \end{array}$$

and where faces are now just projections.

Awodey-Lumsdaine's inductive definition with hard-wired faces: take the Y_i as the primitive data.

A formal account of Awodey-Lumsdaine's definition of semi-simplicial sets

initial segment of the types of the $(Y_i)_{i<n}$

$$
\begin{array}{lll}
U_n & : & \mathsf{Type}_2 \\
U_0 & \triangleq & \mathsf{Unit} \\
U_{n+1} & \triangleq & \Sigma X : U_n . (F^{n,n}(X) \to \mathsf{Type}_1)
\end{array}
$$

signature of all faces at dimension $< n$ *of* Y_p *(using "long-jump" – covariant – faces)*

$$
\begin{array}{llll}
F^{n,p} & (X : U_n) & : & \mathsf{Type}_1 \\
F^{0,p} & \mathsf{unit} & \triangleq & \mathsf{Unit} \\
F^{n+1,p} & (X, Y) & \triangleq & \Sigma x : F^{n,p}(X) . \Pi d : [p] \to [n] .\, Y(\underline{d}^{n,p,n}(x))
\end{array}
$$

extracting the initial segment of faces of dimension $< n$ *of the face* $d : [p] \to [q]$ *of a* p*-semi-simplex from the initial segment of faces of dimension* $< n$ *of the* p*-semi-simplex itself*

$$
\begin{array}{llll}
\underline{d}^{n,p,q} & (X : U_n)\ (x : F^{n,p}(X)) & : & F^{n,q}(X) \\
\underline{d}^{0,p,q} & \mathsf{unit}\ \mathsf{unit} & \triangleq & \mathsf{unit} \\
\underline{d}^{n+1,p,q} & (X, Y)\ (x, y) & \triangleq & (\underline{d}^{n,p,q}(x), \lambda d' : [q] \to [n] .\, \mathtt{rew}_Y\ \underline{\alpha}^{n,p,q,n}_{d,d'}(x)\ \mathtt{in}\ y(d' \circ d))
\end{array}
$$

composition of face extraction

$$
\begin{array}{llll}
\underline{\alpha}^{n,p,q,r}_{d,d'} & (X : U_n)\ (x : F^{n,p}(X)) & : & (\underline{d' \circ d})(x) \ = \ (\underline{d'} \circ \underline{d})(x) \\
\underline{\alpha}^{0,p,q,r}_{d,d'} & \mathsf{unit}\ \mathsf{unit} & \triangleq & \mathsf{refl} \\
\underline{\alpha}^{n+1,p,q,r}_{d,d'} & (X, Y)\ (x, y) & \triangleq & (\underline{\alpha}^{n,p,q,r}_{d,d'}(x), \overline{\alpha}^{n,p,q,r}_{d,d'}(x)(y))
\end{array}
$$

A formal account of Awodey-Lumsdaine's definition of semi-simplicial sets (continued)

It then remains to take the coinductive limit $\mathsf{SST}_n : U_n \to \mathsf{Type}_2$ of the definition:

$$\frac{S : \mathsf{SST}_n(X)}{\mathsf{this}\, S : F^{n,n}(X) \to \mathsf{Type}_1} \qquad \frac{S : \mathsf{SST}_n(X)}{\mathsf{first}\, S : \mathsf{SST}_{n+1}(X, \mathsf{this}\, S)}$$

$$\mathsf{SST} \triangleq \mathsf{SST}_0(\mathsf{unit})$$

i.e. SST is the type of all infinite tuples $(Y_0, Y_1, Y_2, \ldots)$ representing a semi-simplicial set.

Then, for $S : \mathsf{SST}$, the total spaces are easily defined by:

$$\mathcal{T}_n(S) \;\triangleq\; (\texttt{let}\ (X, Y) \;\triangleq\; \mathsf{first}^n(S)\ \texttt{in}\ \Sigma x : F^{n,n}(X).Y(x))$$

while the interpretation of non-trivial face $d : [p] \to [q]$ is

$$\begin{array}{lll} d\ \ (u : \mathcal{T}_p(S)) & : & \mathcal{T}_q(S) \\ d\ \ (x, z) & \triangleq & (\texttt{let}\ (x', y') \;\triangleq\; \mathsf{fst}^{\,q-p+1}(x)\ \texttt{in}\ y'\, d) \end{array}$$

Faces commute thanks to $\overline{\alpha}$.

Digression: the status of equality in the definition

Setting

$$\overline{d}^{n,p,q}(x)(y) \triangleq \lambda d' : [q] \to [n].\, \mathtt{rew}_Y\ \underline{\alpha}^{n,p,q,n}_{d,d'}(x)\ \mathtt{in}\ y(d' \circ d)\,,$$

the proof $\overline{\alpha}^{n,p,q,r}_{d,d'}(x)(y)$ proves

$$\mathtt{rew}_{\lambda x.\Pi d'':[r]\to[n].Y(\underline{d''}(x))}\ \underline{\alpha}^{n,p,q,r}_{d,d'}(x)\ \mathtt{in}\ (\overline{d' \circ d})(x)(y) = ((\overline{d'}(\underline{d}(x))) \circ (\overline{d}(x)))(y)$$

with both members in type $\Pi d'' : [r] \to [n].\, Y(\underline{d''}^{n,r,n}(\underline{d'}^{n,q,r}(\underline{d}^{n,p,q}(x))))$.

The equality can be obtained by associativity of composition in the category of injective functions, functional extensionality of equality, composition of equality proofs, and, an extra coherence diagram over equality proofs.

In a type theory with strict equality, the latter coherence diagram is an instance of axiom K.

Without strict equality, the extra coherence diagram can be proved by proving a higher coherence diagram about $\overline{\alpha}$ at smaller dimensions, and recursively, so statements about deeper and deeper equality.

In HoTT, this can be cut at level n by working on types of homotopy level n.

On the contrary of the presheaf definitions, equalities are here proved, not assumed, and it is possible to prove (at least as a meta-argument) that the proofs α are made from deductive and inductive reasoning over reflexivity proofs. Hence they are strict.

Extension to simplicial sets

The basic idea is to characterize simplicial sets from their subsets of pure simplices and to inject degeneracies algebraically, i.e. to replace the signature of all faces at dimension $< n$ of Y_p

$$\begin{array}{lllll} F^{n,p} & (X : U_n) & : & \mathsf{Type}_1 \\ F^{0,p} & \mathsf{unit} & \triangleq & \mathsf{Unit} \\ F^{n+1,p} & (X, Y) & \triangleq & \Sigma x : F^{n,p}(X).\Pi d : [p] \to [n].\, Y(\underline{d}^{n,p,n}(x)) \end{array}$$

by

$$\begin{array}{lllll} F^{n,p} & (X : U_n) & : & \mathsf{Type}_1 \\ F^{0,p} & \mathsf{unit} & \triangleq & \mathsf{Unit} \\ F^{n+1,p} & (X, Y) & \triangleq & \Sigma x : F^{n,p}(X).\Pi d : [p] \to [n].\, [Y(\underline{d}^{n,p,n}(x)) \vee S^n(X)(\underline{d}^{n,p,n}(x))] \end{array}$$

with $S^n(X)(x)$ algebraically characterizing the degenerate n-simplices whose faces are x.

Extension to simplicial sets (continued)

A degenerate n-simplex can canonically be expressed as a pure p-simplex of dimension $p < n$ along some formal (non-trivial) degeneracy map $s : [p] \to [n]$.

For a degenerate n-simplex to have faces x, those faces have to satisfy the degeneracy/face laws induced from s. So, we can informally set:

$$
\begin{array}{lcl}
S^n(X)(x) & : & \mathsf{Type}_1 \\
S^n(X)(x) & \triangleq & \Sigma \left[\begin{array}{l} p < n \\ s : [p] \to [n] \\ (X', Y') \triangleq \mathsf{fst}^{\,n-p}(X) \\ x' : F^{p,p}(X') \\ z' : Y'(x') \end{array} \right] . \, \Pi \left[\begin{array}{l} p' < n \\ d : [n] \to [p'] \end{array} \right] . \, \widehat{s'}^{r,p'}(\widetilde{d'}^{p,r}(x', z')) = \widehat{d}^{n,p'}(\mathsf{fst}^{\,n-p'}(x))
\end{array}
$$

where $d' : [p] \to [r]$ are $s' : [r] \to [p']$ are canonical such that $d \circ s = s' \circ d'$ and $\widehat{d}$, $\widehat{s}$, $\widetilde{d}$ are appropriate semantic interpretations of d and s

However, we would need well-founded induction and we did not succeed to combine well-founded with the dependent structure of the construction.

Extension to simplicial sets (continued)

We instead define $S^n(X)(x)$ by induction on n:

$$
\begin{array}{llll}
D^{n,p} & (X : Un) & : & \mathsf{Type}_1 \\
D^{0,p} & \mathsf{unit} & : & \mathsf{Empty} \\
D^{n+1,p} & (X, Y) & : & D^{n,p}(X) \vee \bigvee \begin{bmatrix} s : [n] \to [p] \\ x : F^{p,p}(X) \\ z : Y(x) \end{bmatrix}.
\end{array}
$$

$$
\begin{array}{llll}
S^n & X\ x & \triangleq & \Sigma s : D^{n,n}(X).\ \Pi \begin{bmatrix} p' < n \\ d : [n] \to [p'] \end{bmatrix}.\ \mathtt{connect\text{-}top}^{n,n}(X)(x)(s)(d)
\end{array}
$$

with $\mathtt{connect\text{-}top}^{n,p}$ defined by induction on n and expressing that the appropriate constraint on the face d of the degeneracy s holds.

Excerpt of the definition of `connect-top`

See paper.

The coinductive limit

As for semi-simplicial sets, we can define the coinductive limit $\mathsf{ST}_n : U_n \to \mathsf{Type}_2$ of the definition:

$$\frac{S : \mathsf{ST}_n(X)}{\mathsf{this}\, S : F^{n,n}(X) \to \mathsf{Type}_1} \qquad \frac{S : \mathsf{ST}_n(X)}{\mathsf{first}\, S : \mathsf{ST}_{n+1}(X, \mathsf{this}\, S)}$$

$$\mathsf{ST} \triangleq \mathsf{ST}_0(\mathsf{unit})$$

i.e. ST is the type of all infinite tuples $(Y_0, Y_1, Y_2, ...)$ characterizing a simplicial set from the family of its pure simplices dependent over their faces.

For $S : \mathsf{ST}$, the total spaces are now defined including the degeneracies:

$$\mathcal{T}_n(S) \;\triangleq\; (\mathtt{let}\ (X, Y) \;\triangleq\; \mathsf{first}^n(S)\ \mathtt{in}\ \Sigma x : F^{n,n}(X).[Y(x) \vee S^n(X)(x)])$$

Faces are defined as for semi-simplicial sets but defining degeneracies require much more work, and in particular to show coherence diagrams (not all details are proven yet).

Miscellaneous comments

The inductive definition is only classically equivalent to the presheaf definition (classical logic is needed to decide whether a simplex is degenerate or not).

Morphisms can be defined by induction, following the structure of the definition (not done in details).

Similarly for products of simplicial sets (using a characterization of when pairs of simplices are degenerate, but not done in details).

The question of exponentials is open (the characterization of when a simplex in the exponential is degenerate is intricate; afaicj proving that the exponential is an exponential would require classical reasoning to decide degeneracy while curryfying).

This directly scales to a construction of Reedy presheaves (and in particular to cubical sets).